Your Name:	Your Signature:

- **Exam duration:** 3 hours and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything. You are allowed only one A4-sized sheet of paper.
- Solve whatever problems you can solve. The exam is long but (mostly) straightforward. If you've been studying, you'll find it a breather. If you haven't, then I hope the agony is bearable.
- **Simple, non-programmable calculators** are allowed.
- In order to receive credit, you must **show all of your work**.
- If you need more room, use the back of the pages and indicate that you have done so.
- This exam has 24 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules. Good luck! I hope you do well.

Question Number	Maximum Points	Your Score
1	15	
2	20	
3	35	
4	30	
Total	100	

1. (15 points) Solve the following unrelated problems on truncation error, round off errors, Taylor series approximation, error bounds, and basic Matlab commands.

   (a) (5 points) Given a function $f(x) = xe^x + x$, compute the second order Taylor series approximation $f_{\text{quad}}(x)$ around $x_0 = 0$ then using the remainder form and error bound, compute an upper bound on $|f(x) - f_{\text{quad}}(x)|$ for $x \in [0, 1]$.

(b) (5 points) Given these two numbers: $a = 0.54617$ and $b = 0.54601$, and using 4-digit arithmetic to approximate $c = a + b$ and $d = a - b$ then determine the absolute and relative errors using *(i)* chopping and *(ii)* rounding. Tabulate the results.

(c) (5 points) Produce the output of these Matlab commands (1)–(4):

```
(1) linspace(-2, 3, 7);
(2) polyval([1 -3 2], [2 3])
(3) (ones(2,4)-eye(2,4))*diag(2:5)*(ones(4,2)-eye(4,2))
(4) A=[1 2;3 4];B=[1 0;0 1]; disp(A.*B); disp(A*B)
```

2. (20 points) Solve the following unrelated problems on computing solutions for nonlinear system of equations, fixed point iteration, and convergence of numerical methods.

   (a) (5 points) In his famous song *What's My Name?* with Rihanna, Drake sings *the square root of 69 is 8-some* meaning he is approximating the square root of 69 to be 8-point-something. Anyway, apply two iterations each of Newton's method and the bisection method to find the square root of 69 ($f(x) = x^2 - 69$) starting with an initial guess x_0 that is 8-some. For the bisection method, you can select another smart initialization for the bracket of your choice. Tabulate your results.

(b) (5 points) **[Tricky Problem]** Let's revisit the example we discussed in class on the fixed point iteration for $f(x) = x^3 + 4x^2 - 10 = 0$. We know this function has a fixed point for $x \in [1,2]$. We saw in class how this function can be re-written in terms of the fixed point iteration $x = g(x)$. Consider two candidate fixed point iteration functions:

$$g_1(x) = \sqrt{\frac{10}{4+x}}\ ,\ g_2(x) = \frac{1}{2}\sqrt{10 - x^3}.$$

Do these two functions result in convergent fixed point iterations? If so, which one converges faster, i.e., which one has provably a smaller rate of change K? You can make sensible approximations for this problem and also assume a sub-interval within $x \in [1,2]$ if that helps you.

(c) (10 points) The concentration of pollutant bacteria $c(t)$ evolves according to the decaying exponential, given as follows

$$c(t) = 75e^{-1.5t} + 20e^{-0.075t}.$$

This closed-form expression for the concentration is derived from a simple ODE. Implement two iterations of the secant and the Newton-Raphson methods to find the time t that results in $c(t) = 9$. You can start with an initial guess $t_0 = 6$. For the secant method, you can pick $t_1 = 12$. Tabulate your results.

3. (35 points) Solve the following unrelated problems on linear algebra background.
   (a) (5 points) For what values of α will this system of equations

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \alpha \\ 0 & 0 & \alpha - 4 \end{bmatrix} x = \begin{bmatrix} \alpha \\ \alpha - 3 \\ \alpha \end{bmatrix}$$

have no solutions, a unique solution, or infinitely many solutions?

(b) (5 points) Compute the one-norm $||A||_1$, two-norm $||A||_2$, and infinity norm $||A||_\infty$ of this matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$.

(c) (5 points) Compute the nuclear norm, the Frobenius norm, eigenvalues, and eigenvectors of $A = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$ then write the diagonal transformation of matrix $A = TDT^{-1}$ assuming it is has distinct eigenvalues.

(d) (5 points) Compute the absolute $\kappa_a(x)$ and relative $\kappa_r(x)$ condition numbers for this single-valued function $f(x) = e^{-2x}$ and discuss what happens to the condition numbers as x changes (i.e., when does the function become ill-conditioned and well-conditioned).

(e) (5 points) Solve the following linear system of equations

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

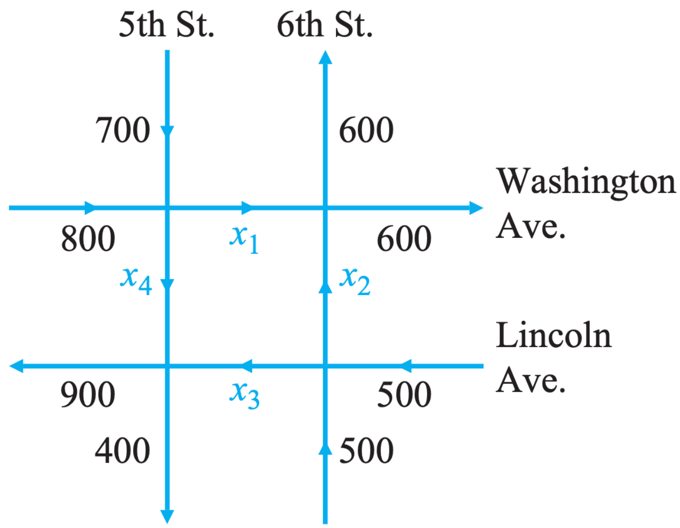
via computing the right pseudo inverse $A^{\dagger}$.

(f) (5 points) **[Tricky Problem]** Compute the relative condition number of this function $f(x_1, x_2) = x_1^{x_2}$ where $x_{1,2} \in \mathbb{R}_+$ (i.e., variables are strictly non-negative) and discuss what happens to the condition number as x changes (i.e., when does the function become ill-conditioned and well-conditioned). You can use the infinity- or the one-norm in your evaluation of the relative condition number.

Hint: Remember that if $h(x) = \alpha^{g(x)}$ then $h'(x) = \ln(\alpha)\alpha^{g(x)}g'(x)$.

(g) (5 points) Given this matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 21 \\ 1 & 11 & 1 \end{bmatrix}$, find its LU decomposition without partial pivoting and then compute x for $Ax = b = \begin{bmatrix} 2 \\ 11 \\ -1 \end{bmatrix}$.

4. (30 points) The rush-hour traffic flow for a network of four one-way streets in a city is shown in the figure below. The numbers next to each street indicate the number of vehicles per hour that enter and leave the network on that street. The variables $x_{1,2,3,4}$ represent the flow of traffic between the four intersections in the network, as shown below:



   (a) (5 points) For a smooth traffic flow, the number of vehicles entering each intersection should always equal the number leaving. For example, since 1,500 vehicles enter the intersection of 5th Street and Washington Avenue each hour and hence $x_1 + x_4 = 1500$ vehicles leave this intersection.

   Determine the linear system of equations determined by the traffic flow at each of the other three intersections, resulting in $Ax = b$. Write down A and b.

(b) (5 points) **Manually** solve this system of equation for $x_{1,2,3,4}$. Note that you *might* get a free variable that can be arbitrarily selected. Verify that your solution satisfies $Ax - b = 0$.

(c) (5 points) Formulate the Richardson's method iteration by decomposing $A = M - N = I_4 - N$ where I_4 is the identity matrix of size $n = 4$. Compute two iterations of the Richardson's method (**with no scaling** α) starting with $x_0 = \begin{bmatrix} 1000 \\ 0 \\ 950 \\ 250 \end{bmatrix}$.

(d) (5 points) Do you need a scaling factor α to guarantee that the Richardson's iteration converges?

Hint: Two of the four eigenvalues of matrix $I_4 - A$ are $\lambda_1 = 0$ and $\lambda_2 = -1$.

(e) (5 points) Formulate the iterations and matrices D, E, F (and hence M and N) for Jacobi, Gauss-Seidel, and SOR methods. Place all these matrices in a table. For the SOR method, choose $\omega = 1.2$.

(f) (5 points) Compute two iterations of the Jacobi, Gauss-Seidel, and SOR methods starting with $x_0 = \begin{bmatrix} 1000 \\ 0 \\ 950 \\ 250 \end{bmatrix}$. Any conclusions?