Your Name: | Your Signature:

- **Exam duration:** 3 hours and 1 minute.
- This exam is closed book, closed notes, closed pretty much everything.
- You will get a solid zero if I catch you cheating or using any form of technology besides a simple calculator.
- **Simple, non-programmable calculators** are allowed. Calculators that compute derivatives and can solve nonlinear equations are NOT allowed.
- Solve whatever problems you can solve. The exam is long but not difficult.
- In order to receive credit, you must **show all of your work**.
- You get to ask me **at most three questions**.
- I hope you do well.

Question Number	Maximum Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (10 points) **Rounding and Chopping Errors**

   Consider two measurements of a structural beam length: $L_1 = 12.568$ m and $L_2 = 12.542$ m. We want to compute the difference $\Delta L = L_1 - L_2$.

   (a) Calculate the exact difference ΔL.

   (b) Compute the difference using **4-digit arithmetic with chopping**. Calculate the absolute error of this result.

   (c) Compute the difference using **4-digit arithmetic with rounding**. Calculate the absolute error of this result.

   (d) What did you notice about the error in the chopping method versus the rounding method?

Solution.

(a) Exact: $\Delta L = 12.568 - 12.542 = 0.026$.

(b) 4-digit Chopping:

$$fl(L_1) = 0.1256 \times 10^2, \quad fl(L_2) = 0.1254 \times 10^2$$

$$\Delta L_{chop} = 0.1256 \times 10^2 - 0.1254 \times 10^2 = 0.0002 \times 10^2 = 0.020$$

Absolute Error: $|0.026 - 0.020| = 0.006$.

(c) 4-digit Rounding:

$$fl(L_1) = 0.1257 \times 10^2, \quad fl(L_2) = 0.1254 \times 10^2$$

$$\Delta L_{round} = 0.1257 \times 10^2 - 0.1254 \times 10^2 = 0.0003 \times 10^2 = 0.030$$

Absolute Error: $|0.026 - 0.030| = 0.004$.

(d) The rounding method produced a smaller error (closer to the true value) compared to chopping.

2. (10 points) **Taylor Series Approximation**

The strength of concrete $f(t)$ (in Mega Pascal or MPa) increases with time t (in days) according to the model:

$$f(t) = 30(1 - e^{-0.1t})$$

We want to approximate the strength near $t_0 = 10$ days.

(a) Compute the first-order approximation $f_1(t)$ (linear) around $t_0 = 10$.

(b) Compute the second-order approximation $f_2(t)$ (quadratic) around $t_0 = 10$.

(c) What does the first derivative $f'(10)$ represent physically in this context?

Solution. Derivatives:

$$f(t) = 30 - 30e^{-0.1t} \implies f(10) = 30(1 - e^{-1}) \approx 30(0.632) = 18.96$$

$$f'(t) = -30(-0.1)e^{-0.1t} = 3e^{-0.1t} \implies f'(10) = 3e^{-1} \approx 1.10$$

$$f''(t) = 3(-0.1)e^{-0.1t} = -0.3e^{-0.1t} \implies f''(10) = -0.3e^{-1} \approx -0.11$$

(a) $f_1(t) = f(10) + f'(10)(t - 10) = 18.96 + 1.10(t - 10)$.

(b) $f_2(t) = 18.96 + 1.10(t - 10) + \frac{-0.11}{2}(t - 10)^2 = 18.96 + 1.10(t - 10) - 0.055(t - 10)^2$.

(c) The first derivative represents the rate at which the concrete is gaining strength (MPa/day) at day 10.

3. (10 points) **Taylor Series Error Bound**

   Using the concrete strength function from the previous problem $f(t) = 30(1 - e^{-0.1t})$, we want to analyze the error of our approximations for the interval $t \in [10, 12]$.

   (a) Find a rigorous upper bound on the error of the **first-order** approximation $|R_1(t)|$ for $t \in [10, 12]$.

   (b) Find a rigorous upper bound on the error of the **second-order** approximation $|R_2(t)|$ for $t \in [10, 12]$.

   Hint. Remember that the function e^{-x} is always *decreasing*.

   (c) Compare the two error bounds. What do you notice about the magnitude of the error as we increase the order of the Taylor series?

Solution.

(a) For $R_1(t)$, we need $f''(\xi)$. $f''(t) = -0.3e^{-0.1t}$. Max $|f''(\xi)|$ on $[10, 12]$ occurs at $\xi = 10$ (decreasing function). Max $|f''| = |-0.3e^{-1}| \approx 0.1104$. Max $(t - 10)^2$ at $t = 12$ is $2^2 = 4$.

$$|R_1(t)| \leq \frac{0.1104}{2!}(4) = 0.2208 \text{ MPa}$$

(b) For $R_2(t)$, we need $f'''(\xi)$. $f'''(t) = -0.3(-0.1)e^{-0.1t} = 0.03e^{-0.1t}$. Max $|f'''(\xi)|$ on $[10, 12]$ occurs at $\xi = 10$. Max $|f'''| = 0.03e^{-1} \approx 0.01104$. Max $(t - 10)^3$ at $t = 12$ is $2^3 = 8$.

$$|R_2(t)| \leq \frac{0.01104}{3!}(8) = \frac{0.01104}{6}(8) = 0.0147 \text{ MPa}$$

(c) The error bound for the second-order approximation (0.0147) is significantly smaller than the first-order bound (0.2208). Increasing the order of the polynomial drastically reduces the error.

4. (10 points) **Numerical Differentiation**

Traffic engineers measured the density of cars $k(t)$ (vehicles/km) on a highway over a one-hour period. The data is as follows, and notice that the step size is $h = 10$.

Time t (min)	0	10	20	30	40	50	60
Density $k(t)$	5	12	25	40	55	65	70

(a) Sketch a rough plot of the data $k(t)$ vs t. Analyze it.

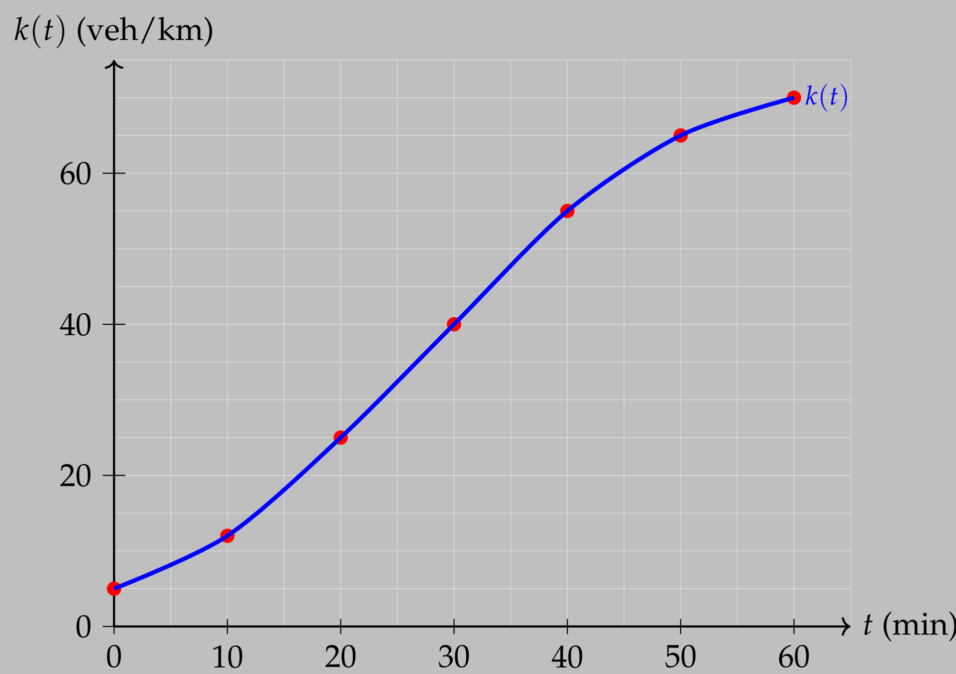
(b) Estimate the rate of change of density $\frac{dk}{dt}$ at $t = 30$ min using the **central difference**.

(c) Estimate the second derivative $\frac{d^2k}{dt^2}$ at $t = 30$ min using the central difference formula:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

(d) Based on your answer in (c), is the congestion accelerating or decelerating at $t = 30$?

Solution.



(a) The data shows an S-curve (sigmoidal). The rate is slow, then fast, then slow again. It is not constant.

(b) Central Difference for $k'(30)$: $k'(30) \approx \frac{k(40)-k(20)}{2h} = \frac{55-25}{20} = \frac{30}{20} = 1.5$ veh/km/min

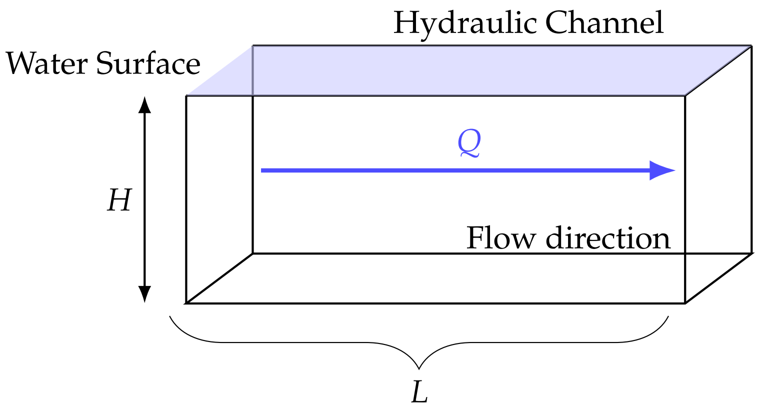
(c) Central Difference for $k''(30)$:

$$k''(30) \approx \frac{k(40) - 2k(30) + k(20)}{h^2} = \frac{55 - 2(40) + 25}{10^2}$$

$$k''(30) \approx \frac{55 - 80 + 25}{100} = \frac{0}{100} = 0 \text{ veh/km/min}^2$$

(d) Since the second derivative is approximately zero, the rate of congestion increase is at its peak (inflection point). It is neither accelerating nor decelerating at this specific moment; the growth is essentially linear locally around $t = 30$.

5. (10 points) **Error Propagation**



The flow rate Q in a specific hydraulic channel (described in the above figure) is approximated by the following equation:

$$Q = C \cdot H^{1.5} \cdot L^{0.5}$$

where H is the hydraulic head and L is the characteristic length. C is a constant coefficient ($C = 2.5$). Suppose we measure the parameters as follows (units omitted for simplicity):

$$H = 4.0 \pm 0.1$$

$$L = 9.0 \pm 0.2$$

Using first-order error propagation, calculate the nominal Flow Rate Q and its absolute error estimate ΔQ.

Solution. Nominal Value: $Q = 2.5(4.0)^{1.5}(9.0)^{0.5} = 2.5(8)(3) = 60.0$.

Error Formula:

$$\Delta Q \approx \left|\frac{\partial Q}{\partial H}\right| \Delta H + \left|\frac{\partial Q}{\partial L}\right| \Delta L$$

Derivatives:

$$\frac{\partial Q}{\partial H} = C(1.5)H^{0.5}L^{0.5} = 2.5(1.5)(2)(3) = 22.5$$

$$\frac{\partial Q}{\partial L} = CH^{1.5}(0.5)L^{-0.5} = 2.5(8)(0.5)\frac{1}{3} = \frac{10}{3} \approx 3.33$$

Calculation:

$$\Delta Q \approx |22.5|(0.1) + |3.33|(0.2)$$

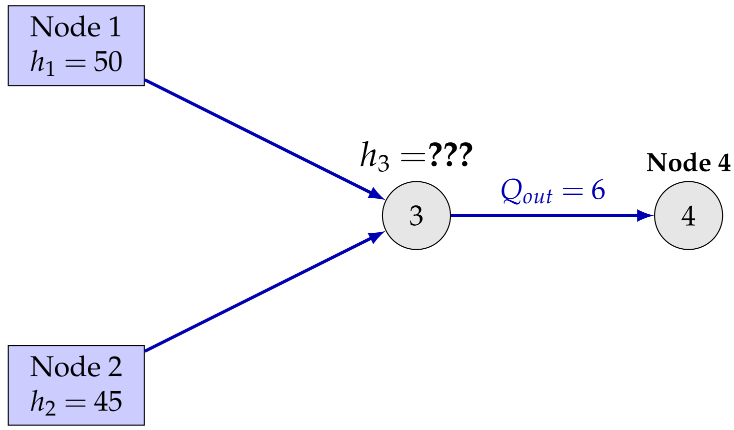
$$\Delta Q \approx 2.25 + 0.67 = 2.92$$

Final Answer: $Q = 60.0 \pm 2.92$.

6. (10 points) **Nonlinear Equations in Hydraulics**

Consider the 4-node pipe network shown below. Nodes 1 and 2 are reservoirs with fixed heads ($h_1 = 50$ m, $h_2 = 45$ m). Node 3 is a junction that feeds into Node 4. The flow continuity at Node 3 ($\sum Q_{in} = Q_{out}$) leads to the following nonlinear equation for the head h_3:

$$\boxed{f(h_3) = \sqrt{50 - h_3} + 2\sqrt{45 - h_3} - 6 = 0.}$$



(a) Formulate the Newton's Method iteration function $h_{i+1} = h_i - \dfrac{f(h)}{f'(h)}$ where h in this equation is basically the unknown variable h_3 that we are trying to solve for and $f(h)$ is is $f(h_3)$.

(b) Starting with an initial guess of $h_{3_0} = 40$ m, perform **two iterations** of Newton's method.

(c) Calculate the approximate relative error after the second iteration.

Solution.

(a) Derivative $f'(h)$:

$$f(h) = (50 - h)^{0.5} + 2(45 - h)^{0.5} - 6$$

$$f'(h) = 0.5(50 - h)^{-0.5}(-1) + 2(0.5)(45 - h)^{-0.5}(-1)$$

$$f'(h) = \frac{-0.5}{\sqrt{50 - h}} - \frac{1.0}{\sqrt{45 - h}}$$

(b) **Iteration 1 ($h_0 = 40$):**

$$f(40) = \sqrt{10} + 2\sqrt{5} - 6 = 3.162 + 2(2.236) - 6 = 3.162 + 4.472 - 6 = 1.634$$

$$f'(40) = \frac{-0.5}{3.162} - \frac{1.0}{2.236} = -0.158 - 0.447 = -0.605$$

$$h_1 = 40 - \frac{1.634}{-0.605} = 40 - (-2.70) = 42.70\,\text{m}$$

Iteration 2 ($h_1 = 42.70$):

$$f(42.70) = \sqrt{50 - 42.7} + 2\sqrt{45 - 42.7} - 6 = \sqrt{7.3} + 2\sqrt{2.3} - 6$$

$$f(42.70) = 2.702 + 2(1.517) - 6 = 2.702 + 3.034 - 6 = -0.264$$

$$f'(42.70) = \frac{-0.5}{2.702} - \frac{1.0}{1.517} = -0.185 - 0.659 = -0.844$$

$$h_2 = 42.70 - \frac{-0.264}{-0.844} = 42.70 - 0.31 = 42.39\,\text{m}$$

(c) **Relative Error:**

$$\epsilon_a = \left|\frac{42.39 - 42.70}{42.39}\right| \times 100\% = \left|\frac{-0.31}{42.39}\right| \times 100\% \approx 0.73\%$$

7. (10 points) **Fixed–Point Iteration and the Role of $g'(x)$**

Consider the nonlinear equation

$$f(x) = x^3 + x - 1 = 0.$$

(a) Rewrite $f(x) = 0$ in the form $x = g(x)$ in three different ways. For each choice, compute $g'(x)$. Then, among your three choices: *(i)* Identify one $g(x)$ such that $|g'(x)| < 1$ for all $x \in [0,1]$. *(ii)*Identify two $g(x)$ for which $|g'(x)| \geq 1$ somewhere in $[0,1]$. Explain what this implies about convergence of the fixed-point iteration $x_{k+1} = g(x_k)$.

Solution.

First FP function. We start from

$$f(x) = x^3 + x - 1 = 0, \quad \to \quad x = 1 - x^3 \quad \to \quad g_1(x) = 1 - x^3 \quad \to \quad g_1'(x) = -3x^2.$$

On $[0,1]$, $|g_1'(x)| = 3x^2$, which exceeds 1 for $x > \frac{1}{\sqrt{3}}$. Hence g_1 is not a contraction on $[0,1]$.

Second FP function:

$$x(1+x^2) = 1 \quad \to \quad x = \frac{1}{1+x^2}, \quad \to \quad g_2(x) = \frac{1}{1+x^2}, \quad \to \quad g_2'(x) = \frac{-2x}{(1+x^2)^2}.$$

For $x \in [0,1]$,

$$|g_2'(x)| = \frac{2x}{(1+x^2)^2}.$$

The maximum occurs at $x = 1$:

$$|g_2'(1)| = \frac{2}{4} = \frac{1}{2} < 1.$$

Thus g_2 is a contraction on $[0,1]$.

Third FP function:

$$x^3 = 1 - x \quad \to \quad x = (1-x)^{1/3} \quad \to \quad g_3(x) = (1-x)^{1/3}, \quad \to \quad g_3'(x) = -\frac{1}{3}(1-x)^{-2/3}.$$

As $x \to 1$, $|g_3'(x)| \to \infty$, so g_3 is not contractive on $[0,1]$.

Fourth FP function:

Add x to both sides of $x^3 + x = 1$ and divide by 2:

$$2x = 1 - x^3 + x \quad \to \quad x = \frac{1 - x^3 + x}{2} \quad \to \quad g_4(x) = \frac{1 - x^3 + x}{2}, \quad \to \quad g_4'(x) = \frac{-3x^2 + 1}{2}.$$

On $[0,1]$,

$$g_4'(x) \in [-1, \tfrac{1}{2}], \quad \to \quad |g_4'(x)| < 1 \quad \text{for } x \in [0,1).$$

Thus g_4 is also contractive on $[0,1)$.

You can in general define

$$g_\alpha(x) = x - \alpha(x^3 + x - 1).$$

Then

$$g'_\alpha(x) = 1 - \alpha(3x^2 + 1).$$

Since $3x^2 + 1 \in [1,4]$ on $[0,1]$, we obtain contraction whenever

$$0 < \alpha < \frac{1}{2}.$$

Conclusion.

Among the rearrangements above:

- g_2 and g_4 are contractions on $[0,1]$ (hence converge).
- g_1 and g_3 are not globally contractive and may diverge.

8. (10 points) **Fixed Point Iteration Theory**

   We are solving for a root using the fixed-point function $g(x) = \sqrt{2-x}$. We are interested in the interval $x \in [0, 1.95]$. The fixed point is exactly $x = 1$.

   (a) Evaluate the derivative function $g'(x)$. Show/calculate the value of $|g'(x)|$ at $x = 1.9$.

   (b) Is the convergence condition $|g'(x)| < 1$ satisfied everywhere in this interval? Plot or sketch the behavior of $|g'(x)|$ briefly to justify.

   (c) Despite your answer in (b), perform 3 iterations starting from $x_0 = 1.75$. Does it appear to converge or diverge?

Solution.

(a) $g(x) = (2-x)^{0.5} \implies g'(x) = \frac{-1}{2\sqrt{2-x}}$. At $x = 1.9$:

$$|g'(1.9)| = \left|\frac{-1}{2\sqrt{0.1}}\right| = \frac{1}{2(0.316)} = \frac{1}{0.632} \approx 1.58$$

(b) No, the condition is not satisfied everywhere. As $x \to 2$, the denominator $\to 0$ and the derivative $\to \infty$. Specifically, for $x > 1.75$, $|g'(x)| > 1$.

(c) Iterations ($x_0 = 1.75$):

$$x_1 = \sqrt{2-1.75} = \sqrt{0.25} = 0.5$$

$$x_2 = \sqrt{2-0.5} = \sqrt{1.5} \approx 1.225$$

$$x_3 = \sqrt{2-1.225} = \sqrt{0.775} \approx 0.88$$

It is oscillating but getting closer to 1 ($|1.75-1| = 0.75 \to |0.5-1| = 0.5 \to |1.225-1| = 0.225$). It is difficult hence to draw a conclusion.

9. (10 points) **Bisection and False Position**

We want to find the root of the function $f(x) = xe^x - 3$.

(a) Determine ***any*** initial bracket $[x_l, x_u]$ of **consecutive integers** that contains the root. Using this bracket, perform **two iterations** of the **bisection method**. Calculate the approximate percent relative error ϵ_a after the second iteration.

(b) Using the same initial bracket, perform **two iterations** of the **false position method**. Calculate the approximate percent relative error ϵ_a after the second iteration.

(c) If we required the absolute error to be strictly less than 10^{-5}, exactly how many iterations (n) would the bisection method require given your initial bracket size? Solve for n.

Solution.

(a) **Bisection Method**

- *Bracket Selection:* $f(1) = 1e^1 - 3 \approx 2.718 - 3 = -0.282$. $f(2) = 2e^2 - 3 \approx 2(7.389) - 3 = 11.778$. Sign change occurs between 1 and 2. Bracket: [1,2].
- *Iteration 1:* $x_{r1} = \frac{1+2}{2} = 1.5$. $f(1.5) = 1.5e^{1.5} - 3 \approx 1.5(4.482) - 3 = 3.723$. $f(1)f(1.5) < 0$ (Negative * Positive), so root is in $[1, 1.5]$.
- *Iteration 2:* $x_{r2} = \frac{1+1.5}{2} = 1.25$.
- *Error:* $\epsilon_a = \left|\frac{1.25-1.5}{1.25}\right| \times 100\% = 20\%$.

(b) **False Position Method** Formula: $x_r = \frac{x_l f(x_u) - x_u f(x_l)}{f(x_u) - f(x_l)}$

- *Iteration 1:* Bracket $[1,2]$. $f(1) = -0.282, f(2) = 11.778$.

$$x_{r1} = \frac{1(11.778) - 2(-0.282)}{11.778 - (-0.282)} = \frac{11.778 + 0.564}{12.06} = \frac{12.342}{12.06} \approx 1.0234$$

$f(1.0234) = 1.0234e^{1.0234} - 3 \approx -0.153$. $f(1.0234)$ is negative. $f(2)$ is positive. New Bracket $[1.0234, 2]$.

- *Iteration 2:* $x_l = 1.0234, f(x_l) = -0.153$. $x_u = 2, f(x_u) = 11.778$.

$$x_{r2} = \frac{1.0234(11.778) - 2(-0.153)}{11.778 - (-0.153)} = \frac{12.053 + 0.306}{11.931} = \frac{12.359}{11.931} \approx 1.0359$$

- *Error:* $\epsilon_a = \left|\frac{1.0359-1.0234}{1.0359}\right| \times 100\% \approx 1.21\%$.

(c) **Theoretical Iterations** Method we learned in Module 2:

$$n > \log_2\left(\frac{1}{0.00001}\right) = \log_2(100{,}000)$$

Since $2^{16} = 65{,}536$ and $2^{17} = 131{,}072$, we need $n = 17$. (Alternatively calculated: $\frac{\ln(100000)}{\ln(2)} \approx 16.6$). Answer: **17 iterations**.

10. (10 points) **Newton's Method (System of Equations)**

We want to find the intersection of a circle and a parabola, described by the system of non-linear equations:

$$u^2 + v^2 - 4 = 0$$
$$u^2 - v - 1 = 0$$

Let the state vector be $\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$.

(a) Determine the Jacobian Matrix $D(u,v) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}$ for this system.

(b) Using the initial guess $\mathbf{x}_0 = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$, compute the first two iterates $\mathbf{x}_1$ and $\mathbf{x}_2$ of the Newton's method. You have to first evaluate the matrix D at $\mathbf{x}_0$ then invert it as we learned in class, then compute $\mathbf{x}_1$ using Newton's iteration then $\mathbf{x}_2$. Remember that Newton's iteration is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - D^{-1}(\mathbf{x}_k) f(\mathbf{x}_k).$$

Solution.

(a) $f_1 = u^2 + v^2 - 4$, $f_2 = u^2 - v - 1$.

$$D = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & 2v \\ 2u & -1 \end{bmatrix}$$

(b) At $u = 1.5, v = 1.0$:

$$f = \begin{bmatrix} (1.5)^2 + 1^2 - 4 \\ (1.5)^2 - 1 - 1 \end{bmatrix} = \begin{bmatrix} 2.25 + 1 - 4 \\ 2.25 - 2 \end{bmatrix} = \begin{bmatrix} -0.75 \\ 0.25 \end{bmatrix}$$

$$D = \begin{bmatrix} 2(1.5) & 2(1.0) \\ 2(1.5) & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & -1 \end{bmatrix}$$

(c) Then

$$\mathbf{x}_1 = \mathbf{x}_0 - D^{-1}(\mathbf{x}_0) f(\mathbf{x}_0) = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix} + \begin{bmatrix} 0.028 \\ 0.333 \end{bmatrix} = \begin{bmatrix} 1.528 \\ 1.333 \end{bmatrix}$$

11. (10 points) **[Bonus Problem] Secant vs. Müller's Method. No partial credit for this problem; it's go big or go home.**

The secant method approximates a nonlinear function using a straight line through two points and estimates the root from the x-intercept. Müller's method instead approximates the function by a quadratic polynomial through three points and takes the quadratic root as the next iterate. I will walk you through this method. Consider

$$f(x) = x^3 - 4 = 0$$

The true root is

$$x^* = \sqrt[3]{4} \approx 1.5874.$$

Suppose we are given

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 1.5.$$

(a) (2 points) Compute $f(x_0), f(x_1), f(x_2)$.

(b) (2 points) Construct the quadratic polynomial

$$p(x) = Ax^2 + Bx + C$$

that satisfies

$$p(x_i) = f(x_i), \quad i = 0,1,2.$$

Write the linear system for A, B, C and solve it. This is a three equations three unknowns system. It's a little annoying to solve but you should be able to do it.

(c) (2 points) Use your quadratic to compute the next Müller iterate by solving

$$Ax^2 + Bx + C = 0.$$

for the root x using the quadratic root equation. You will get two roots. Choose the root closest to x_2. This is basically Müller's method works. This root you identified is now x_3 or the fourth iterate after the initial three data points $x_{1,2,3}$.

(d) (2 points) Now we wanna compare it with the secant method. Compute one secant step using only x_1 and x_2:

$$x_3^{(sec)} = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}.$$

which is the formula we learned in class.

(e) (2 points) Compare the absolute errors

$$|x_3^{(\text{Müller})} - x^*| \quad \text{and} \quad |x_3^{(\text{Secant})} - x^*|.$$

What do you observe?

Solution.

(a) Function values

$$f(1) = 1 - 4 = -3, \quad f(2) = 8 - 4 = 4, \quad f(1.5) = 3.375 - 4 = -0.625.$$

(b) Solve for A, B, C

We solve

$$A + B + C = -3, \quad 4A + 2B + C = 4, \quad 2.25A + 1.5B + C = -0.625.$$

Subtract the first equation from the second and third:

$$3A + B = 7, \quad 1.25A + 0.5B = 2.375.$$

Multiply the second by 2:

$$2.5A + B = 4.75.$$

Now subtract:

$$(3A + B) - (2.5A + B) = 7 - 4.75,$$

hence $A = 4.5$. Substitute into $3A + B = 7$ hence $B = -6.5$. Now use $A + B + C = -3$ to compute $C = -1$. Thus

$$\boxed{p(x) = 4.5x^2 - 6.5x - 1.}$$

(c) Müller iterate

Solve

$$4.5x^2 - 6.5x - 1 = 0.$$

Quadratic formula:

$$x = \frac{6.5 \pm \sqrt{6.5^2 + 18}}{9}.$$

Thus

$$x \approx 1.585, \qquad x \approx -0.14.$$

Choose the root near 1.5:

$$x_3^{(Muller)} \approx 1.585.$$

(d) Secant step

$$x_3^{(Secant)} = 1.5 - \frac{-0.625(1.5-2)}{-0.625-4} = 1.5 - \frac{-0.625(-0.5)}{-4.625} = 1.5 - \frac{0.3125}{-4.625} = 1.5 + 0.0676.$$

Then

$$x_3^{(sec)} \approx 1.568.$$

(e) Compare errors

True root:

$$x^* \approx 1.5874.$$

Müller error:

$$|1.585 - 1.5874| \approx 0.0024.$$

Secant error:

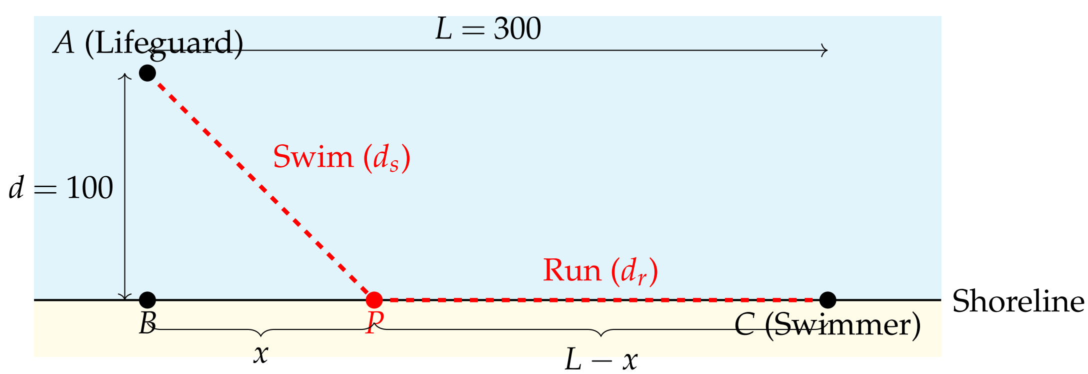
$$|1.568 - 1.5874| \approx 0.0194.$$

The quadratic interpolation produces a significantly smaller error. This illustrates why **Müller's method converges faster than the secant method**.

12. (10 points) **[Bonus Problem 2] Newton's Method: The Lifeguard Problem. No partial credit for this problem; it's go big or go home.**

A lifeguard (Cora?) is positioned in the ocean at point A, which is $d = 100$ meters from the nearest point on the shore (Point B). A drowning swimmer is located at point C on the shoreline, which is a distance of $L = 300$ meters down the coast from Point B. The below figure illustrates this idea.

The lifeguard can swim at a speed of $v_s = 1$ m/s and run on the sand at a speed of $v_r = 3$ m/s. To minimize the total time to reach the swimmer, the lifeguard aims for a point P on the shoreline, x meters from Point B. Point P is the design variable you want to find, so compute this point or find x using Newton's method considering that $x_0 = 50$ meters, as an initial guess. You should compute a few iterations up until you converge to your optimal point P, or x.



Hint: To minimize the Total Time $T(x)$, you must find the x value where the derivative $T'(x) = 0$. Therefore, when applying Newton's Method, your "function" is actually the first derivative $T'(x)$, and the update step requires the second derivative $T''(x)$. You should also use Pythagorean theorem to solve this problem.

Solution.

(a) **Functions and Derivatives:** The total time is Time Swim + Time Run:

$$T(x) = \frac{\sqrt{100^2 + x^2}}{1} + \frac{300 - x}{3} = \sqrt{10000 + x^2} + 100 - \frac{x}{3}$$

First Derivative (set to 0 for minimization):

$$f(x) = T'(x) = \frac{1}{2}(10000 + x^2)^{-1/2}(2x) - \frac{1}{3} = \frac{x}{\sqrt{10000 + x^2}} - \frac{1}{3}$$

Second Derivative (needed for Newton's denominator): Using the quotient rule on $\frac{x}{(10000+x^2)^{1/2}}$:

$$f'(x) = T''(x) = \frac{1 \cdot \sqrt{10000 + x^2} - x \cdot \frac{x}{\sqrt{10000+x^2}}}{10000 + x^2}$$

$$T''(x) = \frac{10000 + x^2 - x^2}{(10000 + x^2)^{3/2}} = \frac{10000}{(10000 + x^2)^{3/2}}$$

(b) **Newton's Update Formula:** Since we are solving $T'(x) = 0$:

$$x_{i+1} = x_i - \frac{T'(x_i)}{T''(x_i)}$$

$$x_{i+1} = x_i - \frac{\frac{x_i}{\sqrt{10000+x_i^2}} - \frac{1}{3}}{\frac{10000}{(10000+x_i^2)^{3/2}}}$$

(c) **Iteration 1 ($x_0 = 50$):** Compute terms at $x = 50$:

$$\sqrt{10000 + 50^2} = \sqrt{12500} \approx 111.803$$

Numerator ($T'(50)$):

$$T'(50) = \frac{50}{111.803} - 0.3333 = 0.4472 - 0.3333 = 0.1139$$

Denominator ($T''(50)$):

$$T''(50) = \frac{10000}{(12500)^{1.5}} = \frac{10000}{1397542.5} \approx 0.007155$$

Update:

$$x_1 = 50 - \frac{0.1139}{0.007155} = 50 - 15.92 = 34.08 \text{ m}$$

(Note: The exact analytical solution is $x = \frac{100}{\sqrt{8}} \approx 35.35$ m, so our first iteration moved very close to the optimal spot).