Your Name: | Your Signature:

- **Exam duration:** 3 hours and 1 minute.
- This exam is closed book, closed notes, closed pretty much everything.
- You will get a solid zero if I catch you cheating or using any form of technology besides a simple calculator.
- **Simple, non-programmable calculators** are allowed. Calculators that compute derivatives and can solve nonlinear equations are NOT allowed.
- Solve whatever problems you can solve. The exam is long but not difficult. Two two bonus problems are difficult so do not attempt them until you are done with the main exam problems.
- In order to receive credit, you must **show all of your work**.
- You get to ask me **at most three questions**.
- I hope you do well.

Question Number	Maximum Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (10 points) **Rounding and Chopping Errors**

   Consider two measurements of a structural beam length: $L_1 = 12.568$ m and $L_2 = 12.542$ m. We want to compute the difference $\Delta L = L_1 - L_2$.

   (a) Calculate the exact difference ΔL.

   (b) Compute the difference using **4-digit arithmetic with chopping**. Calculate the absolute error of this result.

   (c) Compute the difference using **4-digit arithmetic with rounding**. Calculate the absolute error of this result.

   (d) What did you notice about the error in the chopping method versus the rounding method?

2. (10 points) **Taylor Series Approximation**

   The strength of concrete $f(t)$ (in Mega Pascal or MPa) increases with time t (in days) according to the model:

   $$f(t) = 30(1 - e^{-0.1t})$$

   We want to approximate the strength near $t_0 = 10$ days.

   (a) Compute the first-order Taylor series approximation $f_1(t)$ (linear) around $t_0 = 10$.

   (b) Compute the second-order approximation $f_2(t)$ (quadratic) around $t_0 = 10$.

(c) What does the first derivative $f'(10)$ represent physically in this context?

3. (10 points) **Taylor Series Error Bound**

   Using the concrete strength function from the previous problem $f(t) = 30(1 - e^{-0.1t})$, we want to analyze the error of our approximations for the interval $t \in [10, 12]$ and $t_0 = 10$ is the same operating point.

   (a) Find a rigorous upper bound on the error of the **first-order** approximation $|R_1(t)|$ for $t \in [10, 12]$.

   Hint. Remember that the function e^{-x} is always *decreasing* and

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

   where a is the operating point.

(b) Find a rigorous upper bound on the error of the **second-order** approximation $|R_2(t)|$ for $t \in [10, 12]$.

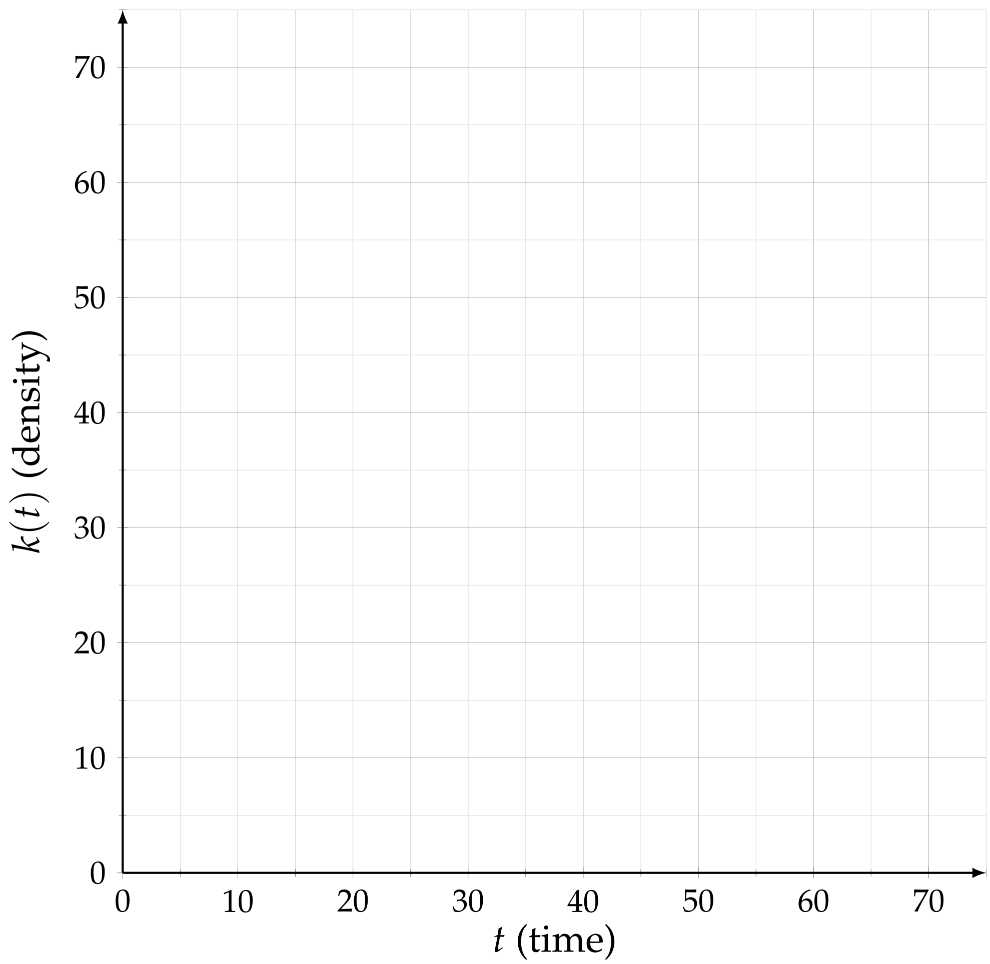
(c) Compare the two error bounds. What do you notice about the magnitude of the error as we increase the order of the Taylor series?

4. (10 points) **Numerical Differentiation**

   Traffic engineers measured the density of cars $k(t)$ (vehicles/km) on a highway over a one-hour period. The data is as follows, and notice that the step size is $h = 10$.

Time t (min)	0	10	20	30	40	50	60
Density $k(t)$	5	12	25	40	55	65	70

   (a) Sketch a rough plot of the data $k(t)$ vs t. Analyze it.



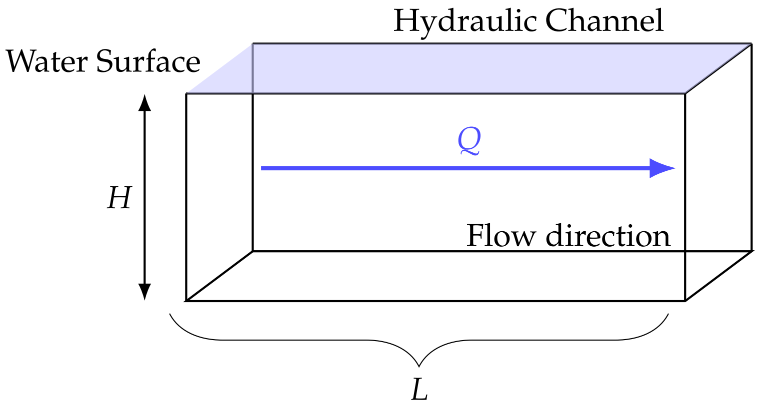
(b) Estimate the rate of change of density $\frac{dk}{dt}$ at $t = 30$ min using the **central difference** approximation we learned in class.

(c) Estimate the second derivative $\frac{d^2k}{dt^2}$ at $t = 30$ min using the central difference formula:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

(d) Based on your answer in (c), is the congestion accelerating or decelerating at $t = 30$?

5. (10 points) **Error Propagation**



The flow rate Q in a specific hydraulic channel (described in the above figure) is approximated by the following equation:

$$Q = C \cdot H^{1.5} \cdot L^{0.5}$$

where H is the hydraulic head and L is the characteristic length. C is a constant coefficient ($C = 2.5$). Suppose we measure the parameters as follows (units omitted for simplicity):

$$H = 4.0 \pm 0.1$$

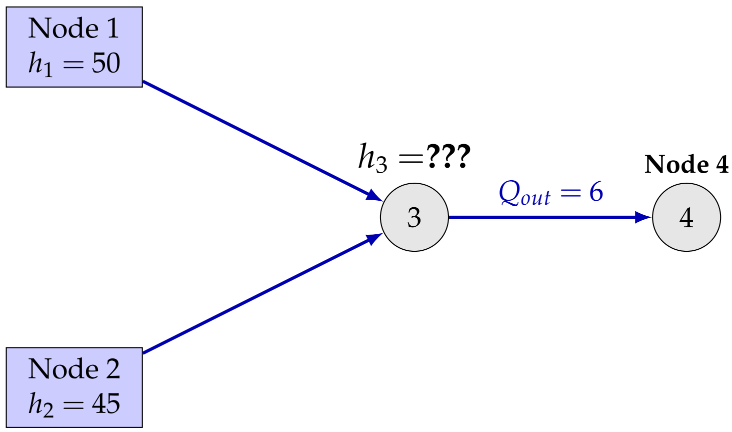
$$L = 9.0 \pm 0.2$$

Using first-order error propagation, calculate the nominal Flow Rate Q and its absolute error estimate ΔQ.

6. (10 points) **Nonlinear Equations in Hydraulics**

Consider the 4-node pipe network shown below. Nodes 1 and 2 are reservoirs with fixed heads ($h_1 = 50$ m, $h_2 = 45$ m). Node 3 is a junction that feeds into Node 4. The flow continuity at Node 3 ($\sum Q_{in} = Q_{out}$) leads to the following nonlinear equation for the head h_3:

$$\boxed{f(h_3) = \sqrt{50 - h_3} + 2\sqrt{45 - h_3} - 6 = 0.}$$



(a) Formulate the Newton's Method iteration function $h_{i+1} = h_i - \dfrac{f(h)}{f'(h)}$ where h in this equation is basically the unknown variable h_3 that we are trying to solve for and $f(h)$ is is $f(h_3)$.

(b) Starting with an initial guess of $h_{3_0} = 40$ m, perform **two iterations** of Newton's method.

(c) Calculate the approximate relative error after the second iteration.

7. (10 points) **FixedPoint Iteration and the Role of** $g'(x)$

   Consider the nonlinear equation

$$f(x) = x^3 + x - 1 = 0.$$

   (a) Rewrite $f(x) = 0$ in the form $x = g(x)$ in three different ways. For each choice, compute $g'(x)$. Then, among your three choices: *(i)* Identify one $g(x)$ such that $|g'(x)| < 1$ for all $x \in [0,1]$. *(ii)*Identify two $g(x)$ for which $|g'(x)| \geq 1$ somewhere in $[0,1]$. Explain what this implies about convergence of the fixed-point iteration $x_{k+1} = g(x_k)$.

8. (10 points) **Fixed Point Iteration Theory**

   We are solving for a root using the fixed-point function $g(x) = \sqrt{2 - x}$. We are interested in the interval $x \in [0, 1.95]$. The fixed point is exactly $x = 1$.

   (a) Evaluate the derivative function $g'(x)$. Show/calculate the value of $|g'(x)|$ at $x = 1.9$.

   (b) Is the convergence condition $|g'(x)| < 1$ satisfied everywhere in this interval? Plot or sketch the behavior of $|g'(x)|$ briefly to justify.

(c) Despite your answer in (b), perform 3 iterations starting from $x_0 = 1.75$. Does it appear to converge or diverge?

9. (10 points) **Bisection and False Position**

   We want to find the root of the function $f(x) = xe^x - 3$.

   (a) Determine ***any*** initial bracket $[x_l, x_u]$ of **consecutive positive integers** (0,1,2,3, etc..) that contains the root.

   Using this bracket, perform **two iterations** of the **bisection method**. Calculate the approximate percent relative error ϵ_a after the second iteration.

(b) Using the same initial bracket, perform **two iterations** of the **false position method**. Calculate the approximate percent relative error ϵ_a after the second iteration.

(c) If we required the absolute error to be strictly less than 10^{-5}, exactly how many iterations (n) would the bisection method require given your initial bracket size? Solve for n.

10. (10 points) **Newton's Method (System of Equations)**

We want to find the intersection of a circle and a parabola, described by the system of non-linear equations:

$$u^2 + v^2 - 4 = 0$$
$$u^2 - v - 1 = 0$$

Let the variable vector be $\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$.

(a) Determine the Jacobian Matrix

$$D(\mathbf{x}) = D(u, v) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} =$$

for this system.

(b) Using the initial guess $\mathbf{x}_0 = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$, compute the first two iterates $\mathbf{x}_1$ and $\mathbf{x}_2$ of the Newton's method. You have to first evaluate the matrix D at $\mathbf{x}_0$ then invert it as we learned in class, then compute $\mathbf{x}_1$ using Newton's iteration then $\mathbf{x}_2$. Remember that Newton's iteration is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - D^{-1}(\mathbf{x}_0) f(\mathbf{x}_0).$$

11. (10 points) **[Bonus Problem] Secant vs. Mllers Method. No partial credit for this problem; it's go big or go home.**

The secant method approximates a nonlinear function using a straight line through two points and estimates the root from the x-intercept. Mllers method instead approximates the function by a quadratic polynomial through three points and takes the quadratic root as the next iterate. I will walk you through this method. Consider

$$f(x) = x^3 - 4 = 0$$

The true root is

$$x^* = \sqrt[3]{4} \approx 1.5874.$$

Suppose we are given

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 1.5.$$

(a) (2 points) Compute $f(x_0), f(x_1), f(x_2)$.

(b) (2 points) Construct the quadratic polynomial

$$p(x) = Ax^2 + Bx + C$$

that satisfies

$$p(x_i) = f(x_i), \quad i = 0,1,2.$$

Write the linear system for A, B, C and solve it. This is a three equations three unknowns system. It's a little annoying to solve but you should be able to do it.

(c) (2 points) Use your quadratic to compute the next Mller iterate by solving

$$Ax^2 + Bx + C = 0.$$

for the root x using the quadratic root equation. You will get two roots. Choose the root closest to x_2. This is basically Mller's method works. This root you identified is now x_3 or the fourth iterate after the initial three data points $x_{1,2,3}$.

(d) (2 points) Now we wanna compare it with the secant method. Compute one secant step using only x_1 and x_2:

$$x_3^{(\text{Secant})} = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}.$$

which is the formula we learned in class.

(e) (2 points) Compare the absolute errors

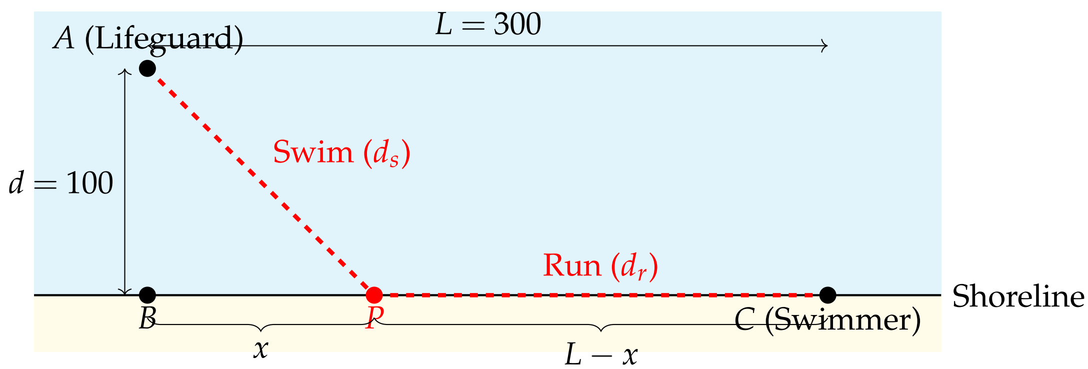
$$|x_3^{(\text{Mller})} - x^*| \quad \text{and} \quad |x_3^{(\text{Secant})} - x^*|.$$

What do you observe?

12. (10 points) **[Bonus Problem 2] Newton's Method: The Lifeguard Problem. No partial credit for this problem; it's go big or go home.**

A lifeguard (Cora?) is positioned in the ocean at point A, which is $d = 100$ meters from the nearest point on the shore (Point B). A drowning swimmer is located at point C on the shoreline, which is a distance of $L = 300$ meters down the coast from Point B. The below figure illustrates this idea.

The lifeguard can swim at a speed of $v_s = 1$ m/s and run on the sand at a speed of $v_r = 3$ m/s. To minimize the total time to reach the swimmer, the lifeguard aims for a point P on the shoreline, x meters from Point B. Point P is the design variable you want to find, so compute this point or find x using Newton's method considering that $x_0 = 50$ meters, as an initial guess. You should compute a few iterations up until you converge to your optimal point P, or x.



Hint: To minimize the Total Time $T(x)$, you must find the x value where the derivative $T'(x) = 0$. Therefore, when applying Newton's Method, your "function" is actually the first derivative $T'(x)$, and the update step requires the second derivative $T''(x)$. You should also use Pythagorean theorem to solve this problem.