# HOMEWORK # 2 SOLUTIONS

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**Due date of this homework: January 27th, 2026 at 11:59:59**.

1. Find the actual error, absolute error, and relative error when approximating $p$ by $\hat{p}$ in each case.

   (a) $p = 0.2500 \times 10^2, \quad \hat{p} = 0.26 \times 10^2$.
   (b) $p = 0.2500 \times 10^{-4}, \quad \hat{p} = 0.26 \times 10^{-4}$.
   (c) $p = 0.2500 \times 10^6, \quad \hat{p} = 0.26 \times 10^6$.

   For each case, compute:

$$\text{actual error} = \hat{p} - p, \qquad \text{absolute error} = |\hat{p} - p|, \qquad \text{relative error} = \frac{|\hat{p} - p|}{|p|}.$$

   After completing the computations, explain in a few sentences what you think the purpose of this problem is. What does it reveal about absolute error versus relative error?

   **Solutions.**

| case | $p$ | $\hat{p}$ | absolute error | relative error |
|---|---|---|---|---|
| 1 | 25 | 26 | 1 | 0.04 |
| 2 | $2.5 \times 10^{-5}$ | $2.6 \times 10^{-5}$ | $1.0 \times 10^{-6}$ | 0.04 |
| 3 | $2.5 \times 10^5$ | $2.6 \times 10^5$ | $1.0 \times 10^4$ | 0.04 |

   **Discussion.** Although the absolute error changes dramatically with scale, the relative error is the same in all three cases. This illustrates that relative error measures accuracy in a scale-independent way, whereas absolute error alone can be misleading when comparing quantities of very different magnitudes.

2. Evaluate $e^{-5}$ via the given two function evaluations (which are equivalent):

$$e^{-x} = \sum_{i=1}^{20} \frac{(-x)^i}{i!} = 1 - x + x^2/2! - x^3/3! + x^4/4! + \cdots$$

   and

$$e^{-x} = \frac{1}{e^x} = \frac{1}{\sum_{i=1}^{20} \frac{(-x)^i}{i!} = 1 - x + x^2/2! - x^3/3! + x^4/4! + \cdots}$$

   and compare with the true value of $6.737947 \times 10^{-3}$. Compute true and approximate relative errors as terms are added and explain what you observe. Plot the errors for the two functions as you add more terms to series.

   **Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

3. Given the two numbers $a = 0.54617$ and $b = 0.54601$, and using four-digit arithmetic approximate $a + b$ and $a - b$. When you are doing so, please note that at each stage of the computation you should only use four-digit arithmetic and not more. Then, once you're done determine the absolute error and relative error using both chopping and rounding. What do you notice?

   **Solutions.** The exact sum and difference are

$$s = p + q = 1.09218, \qquad d = p - q = 0.00016.$$

**Using four-digit chopping arithmetic with floating point (fl) arithmetic:** We first store the operands using chopping:

$$\text{fl}(p) = 0.5461, \qquad \text{fl}(q) = 0.5460.$$

Then

$$\begin{aligned}
\hat{s} = p \oplus q &= \text{fl}\big(\text{fl}(p) + \text{fl}(q)\big) \\
&= \text{fl}(0.5461 + 0.5460) \\
&= \text{fl}(1.0921) \\
&= \text{fl}(0.10921 \times 10^1) \\
&= 0.1092 \times 10^1.
\end{aligned}$$

The absolute and relative errors are

$$|s - \hat{s}| = |1.09218 - 1.09200| = 0.00018,$$

$$\frac{|s - \hat{s}|}{|s|} = \frac{0.00018}{1.09218} \approx 0.0001648.$$

For computing difference, we obtain

$$\begin{aligned}
\hat{d} = p \ominus q &= \text{fl}\big(\text{fl}(p) - \text{fl}(q)\big) \\
&= \text{fl}(0.5461 - 0.5460) \\
&= \text{fl}(0.0001) \\
&= \text{fl}(0.1 \times 10^{-3}) \\
&= 0.1 \times 10^{-3}.
\end{aligned}$$

The absolute and relative errors are

$$|d - \hat{d}| = |0.00016 - 0.00010| = 0.00006,$$

$$\frac{|d - \hat{d}|}{|d|} = \frac{0.00006}{0.00016} = 0.375.$$

**Using four-digit rounding arithmetic with floating point (fl) arithmetic:** We now store the operands using rounding:

$$\text{fl}(p) = 0.5462, \qquad \text{fl}(q) = 0.5460.$$

For sum, we obtain:

$$\begin{aligned}
\hat{s} = p \oplus q &= \text{fl}\big(\text{fl}(p) + \text{fl}(q)\big) \\
&= \text{fl}(0.5462 + 0.5460) \\
&= \text{fl}(1.0922) \\
&= \text{fl}(0.10922 \times 10^1) \\
&= 0.1092 \times 10^1.
\end{aligned}$$

The absolute and relative errors are

$$|s - \hat{s}| = |1.09218 - 1.09200| = 0.00018,$$

$$\frac{|s - \hat{s}|}{|s|} \approx 0.0001648.$$

For computing the difference, we get:

$$\begin{aligned}
\hat{d} = p \ominus q &= \text{fl}\big(\text{fl}(p) - \text{fl}(q)\big) \\
&= \text{fl}(0.5462 - 0.5460) \\
&= \text{fl}(0.0002) \\
&= \text{fl}(0.2 \times 10^{-3}) \\
&= 0.2 \times 10^{-3}.
\end{aligned}$$

The absolute and relative errors are

$$|d - \hat{d}| = |0.00016 - 0.00020| = 0.00004,$$

$$\frac{|d - \hat{d}|}{|d|} = \frac{0.00004}{0.00016} = 0.25.$$

4. Evaluate the polynomial

$$y = x^3 - 7x^2 + 8x + 0.35$$

at $x = 1.37$, using 3-digit arithmetic with chopping. As in the previous problem, when you are adding terms in the above equation you should only be using 3-digit arithmetic.

Evaluate the relative error percentage. Then, when you're done repeat the same problem but given that

$$y = ((x - 7)x + 8)x + 0.35$$

and compare it with the first form of $y$. Show every single step of the computation. Does nesting (i.e., writing the same equation in a different form as in the previous equation) results in a smaller error? Justify your answer.

**Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

5. Determine the number of terms necessary to approximate $\sin(x)$ to 8 significant figures using the Maclaurin series approximation,

$$\sin(x) = x - x^3/3! + x^5/5! + \ldots$$

Calculate the approximation using a value of $x = 0.3\pi$. Automate this with a Matlab code. Your Matlab function should break out of the loop when it detects that 8 significant figures have been reached. By the way, you should utilize the knowledge of the absolute approximation error (or relative error) relative to the number of significant figures to compute an upper bound on the tolerance that you will then use in your code.

**Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

6. Compute the tangent plane (linear) and quadratic approximations of this multivariable function

$$f(x_1, x_2) = \frac{2x_1 + 3}{4x_2 + 1}$$

at the operating point $(0, 0)$, i.e., compute the first order Taylor series approximation of this function then plot the actual function versus its approximations.

**Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

7. Prove that this second order Taylor series approximation

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i) + 0.5f''(x_i)(x_{i+1} - x_i)^2$$

is indeed exact for all values of $x$ if

$$f(x) = ax^2 + bx + c.$$

Why is that true?

**Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

8. Prove Taylor's theorem

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + \underbrace{\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt}_{R_n}$$

given this hint from the fundamental theory of calculus:

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

Integrate by parts and then formulate an inductive reasoning as to why Taylor's theorem is correct. When you are integrating by parts, it will be tricky to evaluate the functions at their boundary interval. That's the tricky part of the problem.

**Solutions.** Gavin please draft the solution to this.

9. **[CEE Application Problem]** In drinking water distribution networks, headloss (lost of pressure/head due to friction) along a pipe is commonly modeled as

$$f(q) = R\, q|q|^{\mu},$$

where

- $q$ is the volumetric flow rate ($\mathrm{m}^3/\mathrm{s}$),
- $f(q)$ is the headloss gradient (m),
- $R > 0$ is a resistance coefficient,
- $\mu \approx 1.85$ for turbulent flow.

This headloss formula is, by the way, mostly empirical (i.e., engineers measured loss of pressure then established this relationship...turns out it is pretty accurate).

Anyway, assume that $R = 5.0 \times 10^6$ and $\mu = 1.85$ then solve the following problems:

(a) For a positive operating flow $q_0 = 0.10\,\mathrm{m}^3/\mathrm{s}$, compute the linear, quadratic, and cubic Taylor polynomial approximations of $f(q)$.

(b) Plot the three approximations versus the actual function and make sure that your plots are clean and clear. Zoom in around $q_0$ or in its neighborhood to observe the accuracy of the approximations. What do you notice?

(c) For each approximation, write the Taylor remainder term and describe the expected approximation error near $q_0$. For this nonlinear headloss model, perhaps you can compute the remainder term via evaluating the integral. If you cannot integrate by parts, ask for Matlab's help.

(d) Which approximation would you use to balance simplicity with accuracy?

(e) Is it possible to construct a Taylor expansion of $f(q)$ about $q_0 = 0$. Explain your answer.

**Solutions.** Gavin please draft the solution to this.

10. **[CEE Application Problem]** Manning's formula for a rectangular channel can be written as

$$Q = \frac{1}{n}\frac{(BH)^{5/3}}{(B+2H)^{2/3}}\sqrt{S}$$

where $Q$ is the flow ($m^3/s$), $n$ is a roughness coefficient, $B$ is width ($m$), $H$ is the depth ($m$), and $S$ is the slope. You are applying this formula to a stream where you know that the width $= 20m$ and the depth $= 0.3m$. Unfortunately, you know the roughness and the slope to only a $\pm 10\%$ precision. That is, you know that the roughness is about 0.03 with a range from 0.027 to 0.033 and the slope is 0.0003 with a range from 0.00027 to 0.00033. Use a first-order error analysis to determine the sensitivity of the flow prediction to each of these two factors. Which one should you attempt to measure with more precision?

**Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

11. Consider the function $f(x) = x^2 - 2x + 4$ on the interval $[-2, 2]$ with a sampling time-step $h = 0.25$. Use the forward, backward, and centered finite difference approximations for the first and second derivatives so as to graphically illustrate which approximation is most accurate. Graph all three first derivative finite difference approximations along with the theoretical, and do the same for the second derivative as well. We studied the first derivative approximation in class, but we did not do the second derivative approximation for the forward, backward, and centered divided difference, so you gotta do some research and figure out the formulae for these.

    **Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.

12. **[CEE Application Problem]** Under laminar flow conditions, the steady-state velocity of a spherical particle settling in a fluid can be computed with Stokes law:

    $$v_x = \frac{g}{18}\left(\frac{\rho_s - \rho_f}{\mu}\right) d^2$$

    where $g$ = gravitational acceleration ($= 981 cm \cdot s^{-2}$), $\rho_s$ and $\rho_f$ are the densities of the particle and the fluid, respectively ($g \cdot cm^{-3}$), $\mu$ is the dynamic viscosity ($g \cdot cm^{-1} \cdot s^{-1}$), and $d$ = the sphere's diameter ($cm$).

    Suppose that you have two types of spherical particles: a phytoplankton cell ($d = 30\mu m$, $\rho_s = 1.027 gcm^{-3}$) and a silt particle ($d = 30\mu m$, $\rho_s = 2.65 g \cdot cm^{-3}$) settling in water ($\mu = 0.013 gcm^{-1}s^{-1}$ and $\rho_f = 0.99973 g \cdot cm^{-3}$).

    For each of these particles, (a) compute the settling velocity and (b) perform a first-order error analysis for the settling velocity. For (b) assume that the parameters $\rho_s$, $\rho_f$, $\mu$, and $d$ have uncertainties of $\pm 2\%$ around their mean values.

    **Solutions.** Solution exists in old homework. Work it through and have it in this document and make sure it's correct.