HOMEWORK # 3

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Due date of this homework: February 15th, 2025 at 11:59:59.

1. What are the number of iterations that are needed to find the root of this equation $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} (i.e., an upper bound on $|x^* - x_n|$) using the bisection method using an initial bracket of $[a, b] = [1, 2]$?

2. Does this fixed point iteration function

$$g(x) = x = \sqrt{\frac{10}{4+x}}$$

defined on $x \in [1, 2]$ converge or diverge? What is the corresponding value for K?

3. The objective of this problem is to find the value of $x^* = 5^{\frac{1}{3}}$ by the fixed-point method we learned about in class. Utilize that the solution of $f(x) = x^3 - 5 = 0$ to the following problems.
   (a) First, introduce two different fixed point functions $g_1(x)$ and $g_2(x)$ that are both convergent for any value of x in [1.2]. Your choice of $g_i(x)$ should be non-trivial.
   (b) Perform 10 iterations for each of the fixed point functions starting with $x_0 = 1.5$, show a plot of $|x_i - x_0|$ and $|x^* - x_i|$ versus the iteration number i. Show your code.
   (c) Which of the fixed point functions $g_1(x)$ are $g_2(x)$ have better convergence speed? Discuss.

4. Using Newton's method for higher dimensions, perform three iterations of the following system of nonlinear equations starting from $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. You are not allowed to use a computer in this example, but you can verify your answers via Matlab. That is, I would like to see you manually inverting the Jacobian matrix and computing the iterations.

$$f(x) = \begin{bmatrix} 4x_1^2 - x_2^2 \\ 4x_1x_2^2 - x_1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5. In this problem, you will use the bisection method to solve

$$f(x) = 2x\cos(2x) - (x+1)^2 = 0, \quad x \in [-1, 0].$$

   (a) Write code for the bisection method that takes, as inputs: the function $f(x)$, the range $[a, b]$, relative error tolerance, and maximum number of iterations. Show your code.
   (b) For a tolerance of `tol` $= 10^{-6}$ and the maximum number of iterations `maxIter` $= 10$, show the resulting root x^* of $f(x)$ then report x_i, $|x^* - x_i|$, $|x^* - x_{i-1}|$, and $|x_i - x_{i-1}|$ for $i \geq 1$ in a nice, table format.

6. Show that the secant method iteration can be written as follows

$$x_i = \frac{f(x_{i-1})x_{i-2} - f(x_{i-2})x_{i-1}}{f(x_{i-1}) - f(x_{i-2})}.$$

7. Consider a function $f(x) = -x^3 - \cos(x)$ and starting initializations $x_0 = -1$ and $x_1 = 0$. Compute x_2 and x_3 via the secant method.

8. In this problem, we wish to compare the performance of the bisection, false position, Newton, and secant methods in finding a root for this function $f(x) = \tan(\pi x) - 6$, which has a root $x^* = \dfrac{\arctan(6)}{\pi} \approx 0.44743$.

   Perform 10 iterations of each of all of the methods, considering that the bisection method starts from this initial bracket $[0, 1]$, the secant method is initialized with $x_0 = 0.4$ and $x_1 = 0.48$ and

Newton's method is initialized with $x_0 = 0.4$. Tabulate your results. By the way, the Excel macro available here https://ctan.org/pkg/excel2latex?lang=en allows you to transform any table to a LaTeXtable. There's also another little function that does the same function on Matlab.

Which method converges faster? How many iterations does it take the bisection method to converge? Show a plot similar to the one on Slide 31/40 in Module 3.

9. As we learned in class, the fixed-point iteration method is defined by the recursive relation $x_{k+1} = g(x_k)$. We seek to find a fixed point x^* such that $x^* = g(x^*)$. In this problem, you will derive the sufficient condition $|g'(x)| < 1$ for this iteration to converge and then apply it to a specific function. I will walk you through the proof. First, let $e_k = x_k - x^*$ represent the error at step k. Solve the following problems.

    (a) Write the expression for the error at the next step, e_{k+1}, in terms of g, x_k, and x^*.

    (b) Apply the **mean value theorem** to the function g on the interval between x_k and x^* to show that:
    $$e_{k+1} = g'(c_k) \cdot e_k$$
    where c_k is some value between x_k and x^*.

    (c) Using the result from step 2, argue why the condition $|g'(x)| < 1$ (for all x in the interval of interest) is sufficient to guarantee that the error approaches zero as $k \to \infty$.

    (d) Now consider the iteration function $g(x) = \frac{x}{2} + \frac{1}{x}$ intended to find the root $x^* = \sqrt{2}$.

        i. Determine the derivative $g'(x)$.

        ii. Based on the condition derived the previous part, find the range of values for x (assume $x > 0$) where this iteration is guaranteed to converge locally. Write a computer code to help you verify your analysis.

        iii. If you choose an initial guess of $x_0 = 1$, will the method converge? Justify your answer using the derivative value at the starting point.

    Solution.

    (a) By definition, the error at the next step is:
    $$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

    (b) The MVT states that for a differentiable function g, there exists a ξ_k between x_k and x^* such that:
    $$\frac{g(x_k) - g(x^*)}{x_k - x^*} = g'(\xi_k)$$
    Multiplying both sides by the denominator $(x_k - x^*)$ which is e_k, we get:
    $$g(x_k) - g(x^*) = g'(\xi_k)e_k \implies e_{k+1} = g'(\xi_k)e_k$$

    (c) Taking the absolute value of both sides:
    $$|e_{k+1}| = |g'(\xi_k)||e_k|$$
    If we assume there is a constant K such that $|g'(x)| \leq K < 1$ for all x in the interval, then:
    $$|e_{k+1}| \leq K|e_k|$$
    Applying this recursively, $|e_n| \leq K^n|e_0|$. Since $K < 1$, $\lim_{n\to\infty} K^n = 0$, implying the error approaches zero.

    (d) The derivative is:
    $$g'(x) = \frac{d}{dx}\left(\frac{x}{2} + x^{-1}\right) = \frac{1}{2} - \frac{1}{x^2}$$
    We require $|g'(x)| < 1$.
    $$\left|\frac{1}{2} - \frac{1}{x^2}\right| < 1$$

This splits into two inequalities:

$$-1 < \frac{1}{2} - \frac{1}{x^2} < 1$$

Right Inequality: $\frac{1}{2} - \frac{1}{x^2} < 1 \implies -\frac{1}{x^2} < \frac{1}{2}$, which is always true for real x.

Left Inequality: $-1 < \frac{1}{2} - \frac{1}{x^2} \implies \frac{1}{x^2} < \frac{3}{2} \implies x^2 > \frac{2}{3} \implies x > \sqrt{\frac{2}{3}} \approx 0.816$.

Thus, the method converges for $x > 0.816$.

For $x_0 = 1$:

$$g'(1) = \frac{1}{2} - \frac{1}{1^2} = -0.5$$

Since $|-0.5| = 0.5 < 1$, the condition is satisfied. The method **will converge**.

10. **[CEE Application Problem]** The pressure drop in a pipe between two junctions m and n (fluid travel from junction m to n) can be computed as

$$\Delta p_{mn} = p_m - p_n = f\frac{L\rho V^2}{2D}$$

where Δp is the pressure drop (Pa), f is the friction factor, L is the length of pipe (m), ρ is the density (kg/m^3), V is the velocity (m/s), and D the pipe diameter.

For turbulent flow, the Colebrook equation provides a model to calculate the friction factor as follows:

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\epsilon}{3.7D} + \frac{2.51}{R\sqrt{f}}\right)$$

where ϵ defines the roughness (m) and R is the Reynolds number $R = \mu^{-1}\rho VD$ with μ being the dynamic viscosity parameter (N S/m^2).

(a) Compute Δp for 0.2m-long pipe given the following parameters

$$\rho = 1.23\text{kg/m}^3, \mu = 1.79 \times 10^{-5}\text{Ns/m}^2, D = 0.006\text{m}, V = 45\text{m/s}, \epsilon = 0.0014\text{mm}.$$

Use a numerical method of your choosing to determine the friction factor. Note that for smooth pipes with $R < 10^5$, a good initial guess can be obtained using the Blasius formula, $f = 0.136/R^{-0.25}$.

(b) Repeat the above problem but for a rougher commercial steel pipe with $\epsilon = 0.045$ mm.

(c) What do you notice?

11. **[CEE Application Problem]** In environmental engineering, the following equation can be used to compute the oxygen concentration level $c(t)$ (mg/L) in a river downstream from a sewage discharge

$$c(t) = 10 - 20(e^{-0.2x(t)} - e^{-0.75x(t)})$$

where $x(t)$ is the distance downstream in kilometers as a function of time t.

(a) Determine the distance downstream where the oxygen level first falls to a reading of 5 mg/L. Determine your answer to a 1% relative error. Note that levels of oxygen below 5 mg/L are generally harmful to game fish such as trout and salmon.
Hint: It is within 2 km of the discharge.

(b) Determine the distance downstream at which the oxygen is at a minimum. What is the concentration at that location?

12. **[CEE Application Problem]** The displacement of a structure is defined by the following equation for a damped oscillation:

$$y(t) = 8e^{-kt}\cos(\omega t)$$

where $k = 0.5$ and $\omega = 3$.

(a) Use the graphical method to make an initial estimate of the time required for the displacement to decrease to 4.

(b) Use newton's method to determine the root to a relative error tolerance of $\epsilon_{re} = 0.01\%$.

(c) Use the secant method to determine the root to a relative error tolerance of $\epsilon_{re} = 0.01\%$.