1. Manually compute the eigenvalues, eigenvectors, determinant, condition number, and rank of these two matrices

$$A = \begin{bmatrix} 1 & \pi \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The computations should be by hand but please verify your answers via Matlab.

Problem 1 Solution:

(a) For matrix A: Compute eig(A):

$$\begin{vmatrix} 1-\lambda & \pi \\ -1 & 4-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(4-\lambda) - (-1)(\pi) = 0$$
$$4 + \lambda^2 - 4\lambda - \lambda + \pi = 0$$
$$\lambda^2 - 5\lambda + 4 + \pi = 0$$
$$\implies \lambda = \frac{5 \pm \sqrt{25 - 4(1)(4+\pi)}}{2(1)} = \frac{5 \pm \sqrt{9 - 4\pi}}{2}$$
$$\implies \text{eig}(A) = \begin{bmatrix} 2.5 + i\sqrt{4\pi - 9} \\ 2.5 - i\sqrt{4\pi - 9} \end{bmatrix} \approx \begin{bmatrix} 2.5 + 0.9442i \\ 2.5 - 0.9442i \end{bmatrix}$$

Compute the eigenvectors of A:

For $\lambda_1 = 2.5 + i\sqrt{4\pi - 9}$:

$$\begin{bmatrix} 1 - 2.5 - i\sqrt{4\pi-9} & \pi \\ -1 & 4 - 2.5 - i\sqrt{4\pi-9} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} = 0$$
$$\implies \begin{cases} \left(-1.5 - i\sqrt{4\pi-9}\right) v_a + \pi v_b = 0 \\ -v_a + \left(1.5 - i\sqrt{4\pi-9}\right) v_b = 0 \end{cases}$$
$$\text{choose } v_b = 1 \implies v_a = 1.5 - i\sqrt{4\pi-9}$$

$$\implies v_1 = \begin{bmatrix} 1.5 - i\sqrt{4\pi-9} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.5 - 0.9442i \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2.5 - i\sqrt{4\pi - 9}$:

$$\begin{bmatrix} 1 - 2.5 + i\sqrt{4\pi-9} & \pi \\ -1 & 4 - 2.5 + i\sqrt{4\pi-9} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} = 0$$
$$\implies \begin{cases} \left(-1.5 + i\sqrt{4\pi-9}\right) v_a + \pi v_b = 0 \\ -v_a + \left(1.5 + i\sqrt{4\pi-9}\right) v_b = 0 \end{cases}$$
$$\text{choose } v_b = 1 \implies v_a = 1.5 + i\sqrt{4\pi-9}$$

$$\implies v_2 = \begin{bmatrix} 1.5 + i\sqrt{4\pi-9} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.5 + 0.9442i \\ 1 \end{bmatrix}$$

Compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 1 & \pi \\ -1 & 4 \end{vmatrix} = (1)(4) - (-1)(\pi) = 4 + \pi$$

Compute the condition number:

$$\kappa(A) = ||A||_2 \left|\left|A^{-1}\right|\right|_2 = \sqrt{\lambda_{max}\left(A^\top A\right)}\sqrt{\lambda_{max}\left(\left(A^{-1}\right)^\top A^{-1}\right)}$$

$$A^\top = \begin{bmatrix} 1 & -1 \\ \pi & 4 \end{bmatrix} \implies A^\top A = \begin{bmatrix} 1 & -1 \\ \pi & 4 \end{bmatrix}\begin{bmatrix} 1 & \pi \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & \pi - 4 \\ \pi - 4 & \pi^2 + 16 \end{bmatrix}$$

$$\begin{aligned}
\det\left(A^\top A - \lambda I_2\right) &= 0 \\
\begin{vmatrix} 2 - \lambda & \pi - 4 \\ \pi - 4 & \pi^2 + 16 - \lambda \end{vmatrix} &= 0 \\
(2 - \lambda)\left(\pi^2 + 16 - \lambda\right) - (\pi - 4)^2 &= 0 \\
2\pi^2 + 32 - 2\lambda - \lambda\pi^2 - 16\lambda + \lambda^2 - (\pi - 4)^2 &= 0 \\
\lambda^2 - \left(18 + \pi^2\right)\lambda + \left(2\pi^2 + 32 - \pi^2 + 8\pi - 16\right) &= 0 \\
\lambda^2 - \left(18 + \pi^2\right)\lambda + \left(\pi^2 + 8\pi + 16\right) &= 0 \\
\lambda^2 - \left(18 + \pi^2\right)\lambda + (\pi + 4)^2 &= 0
\end{aligned}$$

$$\implies \lambda = \frac{18 + \pi^2 \pm \sqrt{(18 + \pi^2)^2 - 4(1)(\pi + 4)^2}}{2(1)} \approx 25.90043, 1.96917$$

$$\implies \lambda_{\max}\left(A^\top A\right) = 25.90043$$

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} 4 & -\pi \\ 1 & 1 \end{bmatrix} = \frac{1}{4 + \pi}\begin{bmatrix} 4 & -\pi \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{4+\pi} & \frac{-\pi}{4+\pi} \\ \frac{1}{4+\pi} & \frac{1}{4+\pi} \end{bmatrix}$$

$$\implies \left(A^{-1}\right)^\top = \begin{bmatrix} \frac{4}{4+\pi} & \frac{1}{4+\pi} \\ \frac{-\pi}{4+\pi} & \frac{1}{4+\pi} \end{bmatrix}$$

$$\implies \left(A^{-1}\right)^\top \left(A^{-1}\right) = \begin{bmatrix} \frac{4}{4+\pi} & \frac{1}{4+\pi} \\ \frac{-\pi}{4+\pi} & \frac{1}{4+\pi} \end{bmatrix}\begin{bmatrix} \frac{4}{4+\pi} & \frac{-\pi}{4+\pi} \\ \frac{1}{4+\pi} & \frac{1}{4+\pi} \end{bmatrix} = \begin{bmatrix} \frac{17}{(4+\pi)^2} & \frac{1-4\pi}{(4+\pi)^2} \\ \frac{1-4\pi}{(4+\pi)^2} & \frac{\pi^2+1}{(4+\pi)^2} \end{bmatrix}$$

$$\begin{aligned}
\det\left(\left(A^{-1}\right)^\top \left(A^{-1}\right) - \lambda I_2\right) &= 0 \\
\begin{vmatrix} \frac{17}{(4+\pi)^2} - \lambda & \frac{1-4\pi}{(4+\pi)^2} \\ \frac{1-4\pi}{(4+\pi)^2} & \frac{\pi^2+1}{(4+\pi)^2} - \lambda \end{vmatrix} &= 0 \\
\left(\frac{17}{(4 + \pi)^2} - \lambda\right)\left(\frac{\pi^2 + 1}{(4 + \pi)^2} - \lambda\right) - \left(\frac{1 - 4\pi}{(4 + \pi)^2}\right)^2 &= 0 \\
0.07104 + \lambda^2 - 0.54644\lambda - 0.05143 &\approx 0 \\
\lambda^2 - 0.54644\lambda + 0.01961 &= 0
\end{aligned}$$

$$\implies \lambda = \frac{0.54644 \pm \sqrt{(-0.54644)^2 - 4(1)(0.01961)}}{2(1)} \approx 0.50782, 0.03862$$
$$\implies \lambda_{\max}\left(\left(A^{-1}\right)^\top \left(A^{-1}\right)\right) = 0.50782$$

$$\begin{aligned}
\kappa(A) &= ||A||_2 \left|\left|A^{-1}\right|\right|_2 \\
&= \sqrt{\lambda_{max}\left(A^\top A\right)}\sqrt{\lambda_{max}\left(\left(A^{-1}\right)^\top A^{-1}\right)} \\
&= \sqrt{25.90043}\sqrt{0.50782}
\end{aligned}$$

$$\implies \kappa(A) \approx 3.6267$$

Compute the rank:

$$\begin{bmatrix} 1 & \pi \\ -1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & \pi \\ 0 & 4+\pi \end{bmatrix} \xrightarrow{R_2/(4+\pi)} \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - \pi R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \text{rank}(A) = 2 - 0 = 2$$

(b) For matrix B: Compute $\text{eig}(B)$:

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)((5-\lambda)(9-\lambda) - 8\cdot 6) - 2(4(9-\lambda) - 7\cdot 6) + 3(4\cdot 8 - 7(5-\lambda)) = 0$$
$$(1-\lambda)\left(45 - 5\lambda - 9\lambda + \lambda^2 - 48\right) - 2(36 - 4\lambda - 42) + 3(32 - 35 + 7\lambda) = 0$$
$$(1-\lambda)\left(\lambda^2 - 14\lambda - 3\right) - 2(-4\lambda - 6) + 3(7\lambda - 3) = 0$$
$$\lambda^2 - 14\lambda - 3 - \lambda^3 + 14\lambda^2 + 3\lambda + 8\lambda + 12 + 21\lambda - 9 = 0$$
$$-\lambda^3 + 15\lambda^2 + 18\lambda + 0 = 0$$
$$-\lambda\left(\lambda^2 - 15\lambda - 18\right) = 0$$
$$\implies \lambda = 0, \frac{15 \pm \sqrt{225 - 4(1)(-18)}}{2(1)} = 0, 7.5 \pm \frac{\sqrt{297}}{2}$$

$$\implies \text{eig}(B) = \begin{bmatrix} 0 \\ 7.5 + \frac{\sqrt{297}}{2} \\ 7.5 - \frac{\sqrt{297}}{2} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 16.11684 \\ -1.11684 \end{bmatrix}$$

Compute the eigenvectors of B:

For $\lambda_1 = 0$:

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0 \to \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & -9 & -12\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0$$

$$\to \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0 \to \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0$$

$$\implies \begin{cases} v_a - v_c = 0\\ v_b + 2v_c = 0\\ v_c = v_c\end{cases}$$

$$\implies \text{choose } v_c = 1 \implies v_b = -2 \implies v_a = 1$$

$$\implies v_1 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

For $\lambda_2 \approx 16.11684$:

$$\begin{bmatrix}-15.11684 & 2 & 3\\4 & -11.11684 & 6\\7 & 8 & -7.11684\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0 \to \begin{bmatrix}1 & -0.13230 & -0.19845\\4 & -11.11684 & 6\\7 & 8 & -7.11684\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0$$

$$\to \begin{bmatrix}1 & -0.13230 & -0.19845\\0 & -10.58764 & 6.79380\\0 & 8.92610 & -5.72769\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0 \to \begin{bmatrix}1 & -0.13230 & -0.19845\\0 & 1 & -0.64167\\0 & 8.92610 & -5.72769\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0$$

$$\to \begin{bmatrix}1 & -0.13230 & -0.19845\\0 & 1 & -0.64167\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0 \to \begin{bmatrix}1 & 0 & -0.28334\\0 & 1 & -0.64167\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_a\\v_b\\v_c\end{bmatrix} = 0$$

$$\implies \begin{cases} v_a - 0.28334v_c = 0\\ v_b - 0.64167v_c = 0\\ v_c = v_c\end{cases}$$

$$\implies \text{choose } v_c = 1 \implies v_b = 0.64167 \implies v_a = 0.28334$$

$$\implies v_2 \approx \begin{bmatrix}0.28334\\0.64167\\1\end{bmatrix}$$

For $\lambda_3 \approx -1.11684$:

$$\begin{bmatrix} 2.11684 & 2 & 3 \\ 4 & 6.11684 & 6 \\ 7 & 8 & 10.11684 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 0.94480 & 1.41721 \\ 4 & 6.11684 & 6 \\ 7 & 8 & 10.11684 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 1 & 0.94480 & 1.41721 \\ 0 & 2.33764 & 0.33116 \\ 0 & 1.38640 & 0.19637 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 0.94480 & 1.41721 \\ 0 & 1 & 0.14166 \\ 0 & 1.38640 & 0.19637 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 1 & 0.94480 & 1.41721 \\ 0 & 1 & 0.14166 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 0 & 1.28337 \\ 0 & 1 & 0.14166 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\implies \begin{cases} v_a + 1.28337 v_c = 0 \\ v_b + 0.14166 v_c = 0 \\ v_c = v_c \end{cases}$$

$$\implies \text{choose } v_c = 1 \implies v_b = -0.14166 \implies v_a = -1.28337$$

$$\implies v_3 \approx \begin{bmatrix} -1.28337 \\ -0.14166 \\ 1 \end{bmatrix}$$

Compute $\det(B)$:

$$\det(B) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(5 \cdot 9 - 8 \cdot 6) - 2(4 \cdot 9 - 7 \cdot 6) + 3(4 \cdot 8 - 7 \cdot 5) = 0$$

Compute the condition number:

$$\text{Since } \det(B) = 0, \text{ the condition number is } \kappa(B) = \infty \text{ or undefined.}$$

Compute the rank:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 \leftarrow -4R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow -7R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow -2R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies \operatorname{rank}(B) = 3 - 1 = 2$$

2. You are given this matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ 4 & 4 & 10 \end{bmatrix},$$

(a) Compute the eigendecomposition $A = TDT^{-1}$.

(b) Compute the 2-norm and the nuclear norm of A.

(c) Show that the kth power of **any diagonalizable** matrix A can be written as

$$A^k = TD^kT^{-1}$$

(d) Show that the matrix exponential $e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ can be written as the kth power of the diagonal matrix D.

Problem 2 Solution:

(a) Find eig(A) and D:

$$\begin{aligned}
\begin{vmatrix} 3-\lambda & 2 & 1 \\ -1 & -\lambda & 0 \\ 4 & 4 & 10-\lambda \end{vmatrix} &= 0 \\
(3-\lambda)\left((-\lambda)(10-\lambda) - 0\right) - 2\left(-1(10-\lambda) - 0\right) + 1(-4+4\lambda) &= 0 \\
(3-\lambda)\left(-10\lambda + \lambda^2\right) - 2(-10+\lambda) + (-4+4\lambda) &= 0 \\
-\lambda^3 + 13\lambda^2 - 28\lambda + 16 &= 0 \\
\lambda^3 - 13\lambda^2 + 28\lambda - 16 &= 0 \\
(\lambda - 1)\left(\lambda^2 - 12\lambda + 16\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\implies \lambda = 1, \frac{12 \pm \sqrt{144 - 4(1)(16)}}{2(1)} &= 1, \frac{12 \pm \sqrt{80}}{2} \\
\implies \text{eig}(A) = \begin{bmatrix} 1 \\ 6 - \sqrt{80}/2 \\ 6 + \sqrt{80}/2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 6 - 2\sqrt{5} \\ 6 + 2\sqrt{5} \end{bmatrix}
\end{aligned}$$

$$\implies D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 - 2\sqrt{5} & 0 \\ 0 & 0 & 6 + 2\sqrt{5} \end{bmatrix}$$

Find eigenvectors:

For $\lambda_1 = 1$:

$$\begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ 4 & 4 & 9 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1/2 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\implies \begin{cases} 2v_a + 2v_b = 0 \\ v_b = v_b \\ v_c = 0 \end{cases}$$

$$\implies \text{choose } v_b = 1 \implies v_a = -1$$

$$\implies v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 6 - 2\sqrt{5} \approx 1.52786$:

$$\begin{bmatrix} 1.47214 & 2 & 1 \\ -1 & -1.52786 & 0 \\ 4 & 4 & 8.47214 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 1.35857 & 0.67928 \\ -1 & -1.52786 & 0 \\ 4 & 4 & 8.47214 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 1 & 1.35857 & 0.67928 \\ 0 & -0.16929 & 0.67928 \\ 0 & -1.43428 & 5.75502 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 1.35857 & 0.67928 \\ 0 & 1 & -4.01252 \\ 0 & -1.43428 & 5.75502 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 1 & 1.35857 & 0.67928 \\ 0 & 1 & -4.01252 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 0 & 6.13057 \\ 0 & 1 & -4.01252 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\implies \begin{cases} v_a + 6.13057v_c = 0 \\ v_b - 4.01252v_c = 0 \\ v_c = v_c \end{cases}$$

$$\implies \text{choose } v_c = 1 \implies v_b = 4.01252 \implies v_a = -6.13057$$

$$\implies v_2 = \begin{bmatrix} -6.13057 \\ 4.01252 \\ 1 \end{bmatrix}$$

For $\lambda_3 = 6 + 2\sqrt{5} \approx 10.47214$:

$$\begin{bmatrix} -7.47214 & 2 & 1 \\ -1 & -10.47214 & 0 \\ 4 & 4 & -0.47214 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & -0.26766 & -0.13383 \\ -1 & -10.47214 & 0 \\ 4 & 4 & -0.47214 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 1 & -0.26766 & -0.13383 \\ 0 & -10.7398 & -0.13383 \\ 0 & 5.07064 & 0.06318 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & -0.26766 & -0.13383 \\ 0 & 1 & 0.01246 \\ 0 & 5.07064 & 0.06318 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\to \begin{bmatrix} 1 & -0.26766 & -0.13383 \\ 0 & 1 & 0.01246 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0 \to \begin{bmatrix} 1 & 0 & -0.13049 \\ 0 & 1 & 0.01246 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

$$\implies \begin{cases} v_a - 0.13049 v_c = 0 \\ v_b + 0.01246 v_c = 0 \\ v_c = v_c \end{cases}$$

$$\implies \text{choose } v_c = 1 \implies v_b = -0.01246 \implies v_a = 0.13049$$

$$\implies v_3 = \begin{bmatrix} 0.13049 \\ -0.01246 \\ 1 \end{bmatrix}$$

Construct T and compute T^{-1} using Gauss-Jordan:

$$T = [v_1 \;\; v_2 \;\; v_3] = \begin{bmatrix} -1 & -6.13057 & 0.13049 \\ 1 & 4.01252 & -0.01246 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & -6.13057 & 0.13049 & 1 & 0 & 0 \\ 1 & 4.01252 & -0.01246 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 6.13057 & -0.13049 & -1 & 0 & 0 \\ 0 & -2.11805 & 0.11803 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 6.13057 & -0.13049 & -1 & 0 & 0 \\ 0 & 1 & -0.05573 & -0.47213 & -0.47213 & 0 \\ 0 & 0 & 1.05573 & 0.47213 & 0.47213 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 6.13057 & 0 & -0.94164 & 0.05836 & 0.12360 \\ 0 & 1 & 0 & -0.44721 & -0.44721 & 0.05279 \\ 0 & 0 & 1 & 0.44721 & 0.44721 & 0.94721 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.80001 & 2.80001 & -0.20003 \\ 0 & 1 & 0 & -0.44721 & -0.44721 & 0.05279 \\ 0 & 0 & 1 & 0.44721 & 0.44721 & 0.94721 \end{array}\right]$$

$$\implies T^{-1} = \begin{bmatrix} 1.80001 & 2.80001 & -0.20003 \\ -0.44721 & -0.44721 & 0.05279 \\ 0.44721 & 0.44721 & 0.94721 \end{bmatrix}$$

Putting everything together:

$$A = TDT^{-1} \approx$$

$$\begin{bmatrix} -1 & -6.13057 & 0.13049 \\ 1 & 4.01252 & -0.01246 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 - 2\sqrt{5} & 0 \\ 0 & 0 & 6 + 2\sqrt{5} \end{bmatrix} \begin{bmatrix} 1.80001 & 2.80001 & -0.20003 \\ -0.44721 & -0.44721 & 0.05279 \\ 0.44721 & 0.44721 & 0.94721 \end{bmatrix}$$

(b) Compute $||A||_2$:

$$||A||_2 = \sqrt{\lambda_{\max}\left(A^\top A\right)}$$

$$A^\top A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 4 \\ 1 & 0 & 10 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ 4 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 9+1+16 & 6+0+16 & 3+0+40 \\ 6+0+16 & 4+0+16 & 2+0+40 \\ 3+0+40 & 2+0+40 & 1+0+100 \end{bmatrix}$$
$$= \begin{bmatrix} 26 & 22 & 43 \\ 22 & 20 & 42 \\ 43 & 42 & 101 \end{bmatrix}$$

$$\begin{vmatrix} 26-\lambda & 22 & 43 \\ 22 & 20-\lambda & 42 \\ 43 & 42 & 101-\lambda \end{vmatrix} = 0$$

$$(26-\lambda)\left((20-\lambda)(101-\lambda) - 42^2\right) - 22\left(22(101-\lambda) - 42\cdot 43\right) + 43\left(22\cdot 42 - 43(20-\lambda)\right) = 0$$

$$(26-\lambda)\left(2020 - 20\lambda - 101\lambda + \lambda^2 - 1764\right) - 22\left(2222 - 22\lambda - 1806\right) + 43\left(924 - 860 + 43\lambda\right) = 0$$

$$(26-\lambda)\left(256 - 121\lambda + \lambda^2\right) - 22\left(416 - 22\lambda\right) + 43\left(64 + 43\lambda\right) = 0$$

$$6656 - 3146\lambda + 26\lambda^2 - 256\lambda + 121\lambda^2 - \lambda^3 - 9152 + 484\lambda + 2752 + 1849\lambda = 0$$

$$-\lambda^3 + 147\lambda^2 - 1069\lambda + 256 = 0$$

$$\implies \lambda \approx \begin{bmatrix} 0.24791 \\ 7.41071 \\ 139.34138 \end{bmatrix} \implies \lambda_{\max}\left(A^\top A\right) \approx 139.34138$$

$$\implies ||A||_2 = \sqrt{139.34138} \approx 11.80429$$

Compute $||A||_*$:

$$||A||_* = \sum_{i=1}^{3} \sigma_i(A) = \sqrt{0.24791} + \sqrt{7.41071} + \sqrt{139.34138} \approx 15.02446$$

(c) Want to show that A^k can be written as TD^kT^{-1}.

$$\begin{aligned}
k = 1 :& A = TDT^{-1} \\
k = 2 :& A^2 = AA = \left(TDT^{-1}\right)\left(TDT^{-1}\right) = TDT^{-1}TDT^{-1} \\
& = TDIDT^{-1} = TDDT = TD^2T \\
k = 3 :& A^3 = AAA = \left(TDT^{-1}\right)\left(TDT^{-1}\right)\left(TDT^{-1}\right) \\
& = TDIDIDT^{-1} = TD^3T^{-1} \\
\ldots &
\end{aligned}$$

This patterns continues for all k.

(d) Want to show that $e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ can be written as the kth power of the diagonal matrix D.

$$\begin{aligned}
e^A &= \sum_{i=0}^{\infty} \frac{A^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{TD^iT^{-1}}{i!} \\
&= \sum_{k=0}^{\infty} \frac{TD^kT^{-1}}{k!} \\
&= T\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)T^{-1}
\end{aligned}$$

3. Compute the determinant of this matrix:

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 1 & \lambda & 1 & 0 \\ 0 & 1 & \lambda & 1 \\ 0 & 0 & 1 & \lambda \end{bmatrix}$$

and try to infer something about the determinant of a generalizable n-by-n matrix similar to the structure of A.

Remember that the determinant of any matrix A is given by this formula:

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + \ldots a_{1n} C_{1n}$$

where a_{ij} is the i, jth entry of A and $C_{ij} = (-1)^{i+j} M_{ij}$ is called the **cofactor** of a_{ij} of the **minor** M_{ij} of a_{ij} which is defined to be the determinant of the $n-1$-by-$n-1$ matrix obtained by deleting the ith row and jth column.

Problem 3 Solution:

Compute the determinant:

$$\begin{aligned} \det(A) &= \lambda \begin{vmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix} + 0 + 0 \\ &= \lambda \left(\lambda \left(\lambda^2 - 1 \right) - 1 (\lambda - 0) + 0 \right) - 1 \left(1 \left(\lambda^2 - 1 \right) - 1 (0) + 0 \right) \\ &= \lambda \left(\lambda^3 - \lambda - \lambda \right) - \left(\lambda^2 - 1 \right) \\ &= \lambda \left(\lambda^3 - 2\lambda \right) - \lambda^2 + 1 \\ &= \lambda^4 - 2\lambda^2 - \lambda^2 + 1 \end{aligned}$$

$$\implies \det(A) = \lambda^4 - 3\lambda^2 + 1$$

This matrix is tridiagonal. In M_{11} of A, a 3×3 matrix, the determinant is calculated:

$$\lambda^3 - 2\lambda$$

For 5×5:

$$\lambda^5 - 4\lambda^3 + 3\lambda$$

For 6×6:

$$\lambda^6 - 5 * \lambda^4 + 6 * \lambda^2 - 1$$

Let A_n represent the $n \times n$ tridiagonal matrix with λ along the main diagonal. And ones on the adjacent diagonals. The trend can be shown to follow:

$$\begin{aligned} \text{Even case: } \det(A_n) &= \sum_{i=0}^{n/2} (-1)^i (n - 2i) \lambda^{2i} \\ \text{Odd case: } \det(A_n) &= \sum_{i=0}^{(n-1)/2} (-1)^i (n - (2i + 1)) \lambda^{2i+1} \end{aligned}$$

If n is even, only terms of λ^i where i is even appear. If n is odd, only the corresponding odd terms appear.

4. Verify that the one-norm given by

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

is indeed a norm on $\mathbb{R}^n$ via proving that it satisfies the three vector norm properties.

Problem 4 Solution:

Prove that the vector 1-norm satisfies the three vector properties given that

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

For all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, we have:

(a) Positive definiteness: $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$.

Since we are taking the sum of absolute values, the total summation cannot be negative, so the 1-norm is never negative. Additionally, the only way to sum positive values to zero is if all the summation values are zero, so the 1-norm is only zero if and only if all its entries are zero. Therefore, the vector 1-norm satisfies the positive definiteness property.

(b) Scaling: $||\alpha x|| = |\alpha| \, ||x||$

Proof:

$$\begin{aligned}
||\alpha x|| &= \sum_{i=1}^{n} |\alpha x_i| \\
&= |\alpha x_1| + |\alpha x_2| + \cdots + |\alpha x_n| \\
&= |\alpha| \left(|x_1| + |x_2| + \cdots + |x_n| \right) \\
&= |\alpha| \sum_{i=1}^{n} |x_i| \\
&= |\alpha| \, ||x||
\end{aligned}$$

(c) Triangle inequality: $||x + y|| \leq ||x|| + ||y||$

Proof:

$$||x + y|| = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

For individual elements x_i and y_i that have the same sign, then

$$|x_i + y_i| = |x_i| + |y_i|$$

For individual elements x_i and y_i that have opposite signs, and assuming $x_i > y_i$ (without loss of generality), then

$$|x_i + y_i| = |x_i| - |y_i| < |x_i| + |y_i|$$

Combining these two cases and summing over all elements (which preserves the inequalities) yields

$$||x + y|| \leq ||x|| + ||y||$$

5. Prove that for all vectors $x \in \mathbb{R}^n$, we have

$$||x||_1 \geq ||x||_2.$$

Problem 5 Solution: By definition, we have:

$$||x||_1 = \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + \cdots + |x_n|$$

$$||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Squaring both sides:

$$||x||_1^2 = (|x_1| + |x_2| + \cdots + |x_n|)^2 \qquad ||x||_2^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

Since squaring the sum of individual positive elements is greater than or equal to summing the individual squares of positive elements due to foiling, we have:

$$||x||_1^2 \geq ||x||_2^2$$

Taking the root preserves the inequality, thus:

$$||x||_1 \geq ||x||_2$$

6. Compute $||A||_F$, $||A||_2$, $||A||_1$ and $||A||_\infty$ of this matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}.$$

Problem 6 Solution: Compute $||A||_F$:

$$||A||_F = \sqrt{4+1+0+1+4+1+0+1+4} = \sqrt{16} = 4$$

Compute $||A||_2$:

$$||A||_2 = \sqrt{\lambda_{\max}\left(A^\top A\right)}$$

$$A^\top = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$A^\top A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Find $\text{eig}\left(A^\top A\right)$:

$$\begin{vmatrix} 5-\lambda & -4 & 1 \\ -4 & 6-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)\left((6-\lambda)(5-\lambda)-0\right)+4\left(-4(5-\lambda)-0\right)+1\left(0-(6-\lambda)\right)=0$$

$$(5-\lambda)\left(30+\lambda^2-11\lambda\right)+4\left(-20+4\lambda\right)+\left(-6+\lambda\right)=0$$

$$150+5\lambda^2-55\lambda-30\lambda-\lambda^3+11\lambda^2-80+16\lambda-6+\lambda=0$$

$$-\lambda^3+16\lambda^2-68\lambda+64=0$$

$$\implies \lambda \approx \begin{bmatrix} 1.3142 \\ 5.0586 \\ 9.6272 \end{bmatrix} \implies \lambda_{\text{max}} = 9.6272$$

$$\implies ||A||_2 = \sqrt{9.6272} \approx 3.1028$$

Compute $||A||_1$:

$$\text{col 1: } 2+1+0=3$$
$$\text{col 2: } 1+2+1=4$$
$$\text{col 3: } 0+1+2=3$$
$$\implies ||A||_1 = \text{ max abs column sum } = 4$$

Compute $||A||_\infty$:

$$\text{row 1: } 2+1+0=3$$
$$\text{row 2: } 1+2+1=4$$
$$\text{row 3: } 0+1+2=3$$
$$\implies ||A||_\infty = \text{ max abs row sum } = 4$$

7. Prove that the Frobenius norm of any matrix A is indeed a legitimate matrix norm by showing that it satisfies the basic matrix norm properties.

Problem 7 Solution: For all $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times m}$, and $\alpha \in \mathbb{R}$, we have:

(a) Positive definiteness: $||A||_F \geq 0$, and $||A||_F = 0$ if and only if $A = 0$.

Since we are taking the sum of absolute values, the total summation cannot be negative, so the Frobenius norm is never negative. Furthermore, we are also squaring the individual elements, which removes negativeness. Additionally, the only way to sum positive values to zero is if all the summation values are zero, so the Frobenius norm is only zero if and only if all its entries are zero. Square-rooting also preserves non-negativity. Therefore, the vector Frobenius norm satisfies the positive definiteness property.

(b) Scaling: $||\alpha A||_F = \alpha\, ||A||_F$

Proof:

$$\begin{aligned}
||\alpha A||_F &= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} \left|\alpha a_{ij}\right|^2} \\
&= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |\alpha^2|\,\left|a_{ij}\right|^2} \\
&= \alpha\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} \left|a_{ij}\right|^2} \\
&= \alpha\, ||A||_F
\end{aligned}$$

(c) Triangle inequality: $||A+B||_F \leq ||A||_F + ||B||_F$

Proof:

For scalars, we know that

$$\left|a_{ij} + b_{ij}\right| \leq \left|a_{ij}\right| + \left|b_{ij}\right|.$$

Squaring each term, then summing over elements of matrices A and B, then taking the square root of each term preserves the inequality, yields (with some rearranging):

$$\begin{aligned}
\left|a_{ij} + b_{ij}\right|^2 &\leq \left|a_{ij}\right|^2 + \left|b_{ij}\right|^2 \\
\sum_{i=1}^{n}\sum_{j=1}^{m}\left|a_{ij} + b_{ij}\right|^2 &\leq \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left|a_{ij}\right|^2 + \left|b_{ij}\right|^2\right) \\
\sum_{i=1}^{n}\sum_{j=1}^{m}\left|a_{ij} + b_{ij}\right|^2 &\leq \sum_{i=1}^{n}\sum_{j=1}^{m}\left|a_{ij}\right|^2 + \sum_{i=1}^{n}\sum_{j=1}^{m}\left|b_{ij}\right|^2 \\
\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left|a_{ij} + b_{ij}\right|^2} &\leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left|a_{ij}\right|^2} + \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left|b_{ij}\right|^2}
\end{aligned}$$

$$\implies ||A+B||_F \leq ||A||_F + ||B||_F$$

8. Investigate the values for the condition number of this two dimensional function

$$f(x_1, x_2) = x_1^5 - x_2^2 - 4$$

via two different norms of your choice.

Problem 8 Solution:

$$f(x_1, x_2) = x_1^5 - x_2^2 - 4, \quad J(x) = \begin{bmatrix} 5x_1^4 \\ -2x_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find $\kappa_{a,2}(x)$ and $\kappa_{r,2}(x)$:

$$\kappa_{a,2}(x) = |J(x)|_2 = \sqrt{\left(5x_1^4\right)^2 + (-2x_2)^2} = \sqrt{25x_1^8 + 4x_2^2}$$

$$\kappa_{r,2}(x) = \frac{|J(x)_2| \, |x|_2}{|f(x)|} = \frac{\sqrt{25x_1^8 + 4x_2^2}\sqrt{x_1^2 + x_2^2}}{\left|x_1^5 - x_2^2 - 4\right|}$$

$\kappa_{r,2}(x)$ is ill-conditioned for $f(x)$ near 0 and for large x_1-values due to the x_1^8 term dominating in the numerator. It is well-conditioned for $|x| < 1$ and for large values of $f(x)$ when the numerator is small.

Find $\kappa_{a,1}(x)$ and $\kappa_{r,1}(x)$:

$$\kappa_{a,1}(x) = ||J(x)||_1 = \left|5x_1^4\right| + |-2x_2| = 5x_1^4 + |2x_2|$$

$$\kappa_{r,1}(x) = \frac{||J(x)_1|| \, ||x||_1}{|f(x)|} = \frac{\left(5x_1^4 + |2x_2|\right)(|x_1| + |x_2|)}{\left|x_1^5 - x_2^2 - 4\right|}$$

$\kappa_{r,2}(x)$ is ill-conditioned for $f(x)$ near 0 and for large x_1-values due to the x_1^5 and $5x_1^4$ terms dominating in the numerator. It is well-conditioned for large values of $f(x)$ and when the numerator is small.

9. You are given the following difference equation (basically a three dimensional differential equation but in discrete-time):

$$x(k+1) = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(k)$$

where $x(k) \in \mathbb{R}^3$ is the state-vector and $u(k) \in \mathbb{R}$ is the control input and k defines time. Your objective is to find a control sequence $(u(0), u(1), \ldots, u(n-1))$ that can drive the system from

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to

$$x(n) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

in the least possible time-steps n. You can start by trying $n = 1$ then $n = 2$, etc... and see what kind of relationship you obtain. This relationship can be formulated by deriving a linear system of equations $Au_{\text{optimal}} = b$ where

$$u_{\text{optimal}} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n) \end{bmatrix}$$

is the optimal control input that drives the states from the initial state $x(0)$ to the desired one $x(n)$. Is this doable? The resulting system of equations will be rectangular and can be solved using pseudo inverses we learned about in class.

Problem 9 Solution: Given:

$$x\,(k+1) = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} x\,(k) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u\,(k)\,, x\,(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, x\,(n) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Try $n = 1$:

$$\begin{aligned} x\,(1) &= Ax\,(0) + Bu\,(0) \\ &= \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u\,(0) \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2u\,(0) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 2u\,(0) \end{bmatrix} = x\,(n) \end{aligned}$$

$$\implies \begin{bmatrix} 0 \\ 0 \\ 2u\,(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\implies \text{impossible}$$

Try $n = 2$:

$$\begin{aligned}
x(2) &= Ax(1) + Bu(1) \\
&= A(Ax(0) + Bu(0)) + Bu(1) \\
&= A^2x(0) + ABu(0) + Bu(1) \\
&= 0 + \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(0) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(1) \\
&= \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} u(0) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(1) \\
&= \begin{bmatrix} 2u(0) \\ 8u(0) \\ 4u(0) + 2u(1) \end{bmatrix} = x(n)
\end{aligned}$$

$$\implies \begin{bmatrix} 2u(0) \\ 8u(0) \\ 4u(0) + 2u(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\implies \begin{cases} 2u(0) = 2 \\ 8u(0) = 2 \\ 4u(0) + 2u(1) = 0 \end{cases}$$

$\implies$ impossible, due to inconsistencies of the value of $u(0)$

Try $n = 3$:

$$\begin{aligned}
x(3) &= Ax(2) + Bu(2) \\
&= A(Ax(1) + Bu(1)) + Bu(2) \\
&= A^2x(1) + ABu(1) + Bu(2) \\
&= A^2(Ax(0) + Bu(0)) + ABu(1) + Bu(2) \\
&= A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) \\
&= 0 + \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(0) + \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(1) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(2) \\
&= \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} u(0) + \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} u(1) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(2) \\
&= \begin{bmatrix} 10 \\ 22 \\ 8 \end{bmatrix} u(0) + \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} u(1) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(2) = x(n)
\end{aligned}$$

$$\begin{cases} 10u(0) + 2u(1) = 2 & (1) \\ 22u(0) + 8u(1) = 2 & (2) \\ 8u(0) + 4u(1) + 2u(2) = 0 & (3) \end{cases}$$

From (1):

$$u(1) = 1 - 5u(0)$$

Substitute (1) into (2):

$$\begin{gathered}22u\left(0\right)+8\left(1-5u\left(0\right)\right)=2\\22u\left(0\right)+8-40u\left(0\right)=2\\-18u\left(0\right)+8=2\\6=18u\left(0\right)\end{gathered}$$

$$\implies u\left(0\right)=\frac{1}{3}$$

Plug $u\left(0\right)$ back into (1):

$$u\left(1\right)=1-5\left(\frac{1}{3}\right)\implies u\left(1\right)=-\frac{2}{3}$$

Plug $u\left(0\right)$ and $u\left(1\right)$ into (3):

$$\begin{gathered}8\left(\frac{1}{3}\right)+4\left(-\frac{2}{3}\right)+2u\left(2\right)=0\\0+2u\left(2\right)=0\end{gathered}$$

$$\implies u\left(2\right)=0$$

It is possible to drive the system from its initial state to the desired final state. The optimal control input uses $n=3$ steps, where the optimal control input is:

$$u_{\text{optimal}}=\begin{bmatrix}u\left(0\right)\\u\left(1\right)\\u\left(2\right)\end{bmatrix}=\begin{bmatrix}1/3\\-2/3\\0\end{bmatrix}$$