Homework # 4

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Due date of this homework: March 5th, 2026 at 11:59:59.

1. Manually compute the eigenvalues, eigenvectors, determinant, condition number, and rank of these two matrices

$$A = \begin{bmatrix} 1 & \pi \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

   The computations should be by hand but please verify your answers via Matlab.

2. You are given this matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ 4 & 4 & 10 \end{bmatrix},$$

   (a) Compute the eigendecomposition $A = TDT^{-1}$.

   (b) Compute the 2-norm and the nuclear norm of A.

   (c) Show that the kth power of **any diagonalizable** matrix A can be written as

$$A^k = TD^kT^{-1}$$

   (d) Show that the matrix exponential $e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ can be written as the kth power of the diagonal matrix D.

3. Compute the determinant of this matrix:

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 1 & \lambda & 1 & 0 \\ 0 & 1 & \lambda & 1 \\ 0 & 0 & 1 & \lambda \end{bmatrix}$$

   and try to infer something about the determinant of a generalizable n-by-n matrix similar to the structure of A.

   Remember that the determinant of any matrix A is given by this formula:

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + \ldots a_{1n}C_{1n}$$

   where a_{ij} is the i, jth entry of A and $C_{ij} = (-1)^{i+j}M_{ij}$ is called the **cofactor** of a_{ij} of the **minor** M_{ij} of a_{ij} which is defined to be the determinant of the $n-1$-by-$n-1$ matrix obtained by deleting the ith row and jth column.

4. Verify that the one-norm given by

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

   is indeed a norm on $\mathbb{R}^n$ via proving that it satisfies the three vector norm properties.

5. Prove that for all vectors $x \in \mathbb{R}^n$, we have

$$||x||_1 \geq ||x||_2.$$

6. Compute $||A||_F$, $||A||_2$, $||A||_1$ and $||A||_\infty$ of this matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}.$$

7. Prove that the Frobenius norm of any matrix A is indeed a legitimate matrix norm by showing that it satisfies the basic matrix norm properties.

8. Investigate the values for the condition number of this two dimensional function

$$f(x_1, x_2) = x_1^5 - x_2^2 - 4$$

via two different norms of your choice.

9. You are given the following difference equation (basically a three dimensional differential equation but in discrete-time):

$$x(k+1) = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(k)$$

where $x(k) \in \mathbb{R}^3$ is the state-vector and $u(k) \in \mathbb{R}$ is the control input and k defines time. Your objective is to find a control sequence $(u(0), u(1), \ldots, u(n-1))$ that can drive the system from

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to

$$x(n) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

in the least possible time-steps n. You can start by trying $n = 1$ then $n = 2$, etc... and see what kind of relationship you obtain. This relationship can be formulated by deriving a linear system of equations $Au_{\text{optimal}} = b$ where

$$u_{\text{optimal}} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n) \end{bmatrix}$$

is the optimal control input that drives the states from the initial state $x(0)$ to the desired one $x(n)$. Is this doable? The resulting system of equations will be rectangular and can be solved using pseudo inverses we learned about in class.