

Due date of this homework: March 20th, 2026 at 11:59:59.

1. You are given this linear system of equations $Ax = b$ as follows

$$2x_1 - x_2 + x_3 = -1, \quad 2x_1 + 2x_2 + 2x_3 = 4, \quad -x_1 - x_2 + 2x_3 = -5.$$

- (a) Applying the Gaussian elimination (Slide 19 in Module 05), compute the matrix inverse A^{-1} then find $x^* = A^{-1}b$.
- (b) Compute the mapping matrices T for the Jacobi and the Gauss-Seidel methods and show that $\rho(T_{\text{jacobi}}) \approx 1.12$ and that $\rho(T_{\text{gauss}}) = 0.5$.
- (c) Confirm the findings of part (b) via showcasing convergence/divergence of these two methods.
2. Problem 9.13:

Solve

$$x_1 + x_2 - x_3 = -3, \quad 6x_1 + 2x_2 + 2x_3 = 2, \quad -3x_1 + 4x_2 + x_3 = 1$$

using with Naive Gauss, Gaussian elimination with. partial pivoting, and Gauss-Jordan without partial pivoting.

3. Problems 10.3 (Chapra, Numerical Methods for Engineers, 8th edition):

Solve the following system of equations $Ax = b$ via the LU decomposition without pivoting:

$$15x_1 + 7x_2 - 4x_3 = -51, \quad 4x_1 - 4x_2 + 9x_3 = 62, \quad 12x_1 - x_2 + 3x_3 = 8$$

then determine the matrix inverse after you complete the reduced row echelon form and confirm that $A^{-1}A = I_3$.

4. Problem 11.27 (Chapra, Numerical Methods for Engineers, 8th edition):

Linear algebraic equations can arise in the solution of differential equations. For example, the following differential equation results from a steady-state mass balance for a chemical in a one-dimensional canal (e.g., concentrations of chlorine in pipes of a water distribution network in a city):

$$0 = D \frac{d^2c}{dx^2} - U \frac{dc}{dx} - kc$$

where $c(x)$ is the concentration as a function of x which is the distance. Parameters D, U, k define the diffusion coefficient, fluid velocity, and a first-order decay rate, respectively. Note that typically the concentrations are a function of space (i.e., location in the pipe) and time, but this model forgoes the time-element and focuses on studying concentrations as a function of space only.

Anyway, convert this differential equation to an equivalent system of simultaneous algebraic equations via using the central divided difference for the derivative (and second derivative), that we learned in previous homework assignments.

Solve these equations from $x = 0$ to $x = 10$ with $\Delta x = 2$, and develop a plot of concentration versus distance. The parameters given are: $D = 2.5, U = 0.75, k = 0.15$, with initial conditions $c(0) = 75$ and boundary conditions $c(10) = 25$. Solve the problem again when $\Delta x = 1$ and $\Delta x = 0.1$. What do you notice?

Plot the graphs of $c(x)$ versus x for different Δx .

5. Problem 12.7 (Chapra, Numerical Methods for Engineers, 8th edition):

Fig. 1 shows the Chloride mass balance for the Great Lakes in terms of the flows. Determine the concentration of chloride in each of the Great Lakes $c_{1,2,3,4,5}$ for the five lakes using the

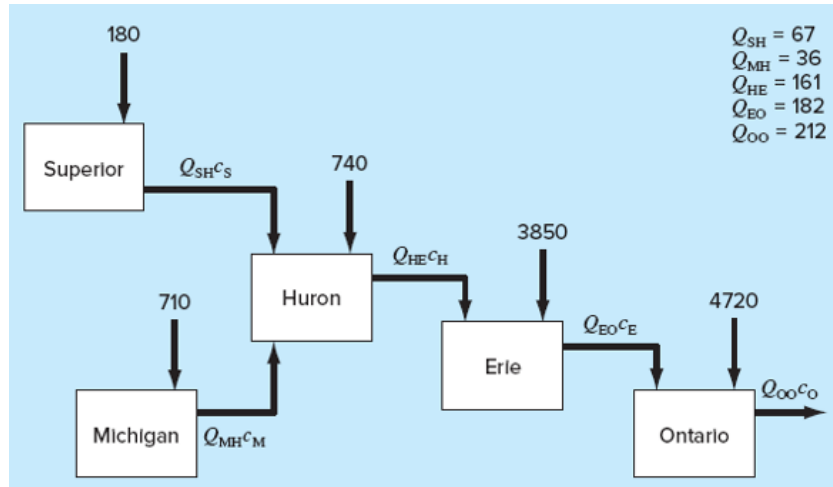


Figure 1: Abstract description of the Chloride mass balance and flows for the Great Lakes.

information shown in Fig. 1. The flow rates are given in the legend of the figure. Compute the matrix inverse, and use it to determine the percent reduction of the concentration in Lake Ontario due to a 50% reduction of the loadings to Lake Superior and Lake Michigan.

To solve the problem, you need to know that the flow rates Q (in cubic meters per minute) and concentrations (in milligrams per cubic meter) are related through simple mass-balance models:

$$\sum q_{in}c_{in} = \sum q_{out}c_{out}.$$

6. Problem 12.21 (Chapra, Numerical Methods for Engineers, 8th edition):

Indoor air pollution deals with air contamination in enclosed spaces such as homes, offices, work areas, etc. Suppose that you are designing a ventilation system for a restaurant as shown in Fig. 2. The restaurant serving area consists of two square rooms and one elongated room. Room 1 and room 3 have sources of carbon monoxide from smokers and a faulty grill, respectively. Steady-state mass balances can be written for each room. For example, for the smoking section (room 1), the balance can be written as

$$0 = \underbrace{W_{smoker}}_{\text{load}} + \underbrace{Q_a c_a}_{\text{inflow}} - \underbrace{Q_a c_1}_{\text{outflow}} + \underbrace{E_{13}(c_3 - c_1)}_{\text{mixing}}$$

or substituting the parameters

$$225c_1 - 25c_3 = 2400.$$

Similar balances can be written for the other rooms.

- Solve for the steady-state concentration of carbon monoxide in each room.
- Determine what percent of the carbon monoxide in the kids' section is due to (i) the smokers, (ii) the grill, and (iii) the air in the intake vents.
- If the smoker and grill loads are increased to 2000 and 5000 mg/hr, respectively, use the matrix inverse to determine the increase in the concentration in the kids' section.
- How does the concentration in the kids' area change if a screen is constructed so that the mixing between areas 2 and 4 is decreased to 5 m³/hr?

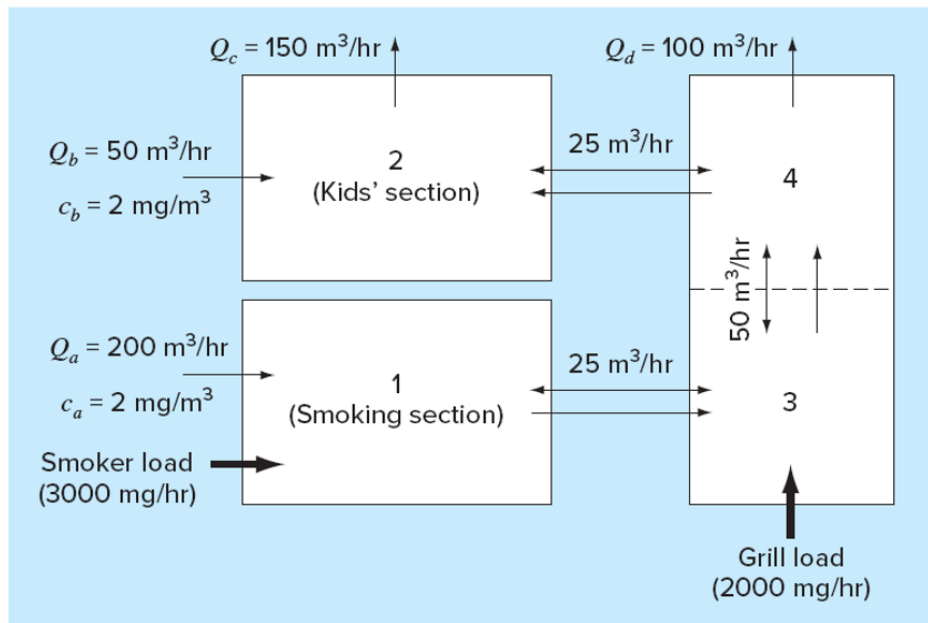


Figure 2: Overhead view of rooms in a restaurant. The one-way arrows represent volumetric airflows, whereas the two-way arrows represent diffusive mixing. The smoker and grill loads add carbon monoxide mass to the system but negligible airflow.

7. Problem 12.22 (Chapra, Numerical Methods for Engineers, 8th edition):

An upward force of 20 kN is applied at the top of a tripod as depicted in Fig. 3. Determine the forces in the legs of the tripod via formulating a linear system of equations $Ax = b$ and solving it via Gaussian elimination **manually** and the SOR method with optimal ω for only 5 iterations. Compare the results.

To solve this problem, you can assume that each leg is in tension, which means that each pulls on point D.

8. For the following system of equations $Ax = b$

$$3x_1 - x_2 + x_3 = 1, \quad 3x_1 + 6x_2 + 2x_3 = 0, \quad 3x_1 + 3x_2 + 7x_3 = 4$$

use the Richardson, Jacobi, Gauss-Seidel, and SOR (with $\omega = 0.5, 1.1, 1.8$) to compute the solution $x^* = A^{-1}b$ for ten iterations. To do so, you need to develop codes for these four methods—you will be using this code in some other problems.

You will also need to show the matrices M and N (D, E, F) for these methods.

Plot the performance of the four methods as a function of the number of iterations and the error norm $\|x^* - x^{(i)}\|$ where i is the iteration number.

9. In this problem, we wish to test the scalability of the iterative methods to solve $Ax = b$.

- (a) Generate a random square matrix A in dimension $n = 10,000$ and a random vector $b \in \mathbb{R}^n$. You can do that via matlab's `randn` command. You have to ensure that the generated matrix is indeed invertible (we talked about how to do so in class) so that the random system of equations $Ax = b$ has a unique solution.

Suggestion: If I were you, I would start with an $n = 5$ or $n = 10$ random system. This makes it easier to debug the code and see what's happening. Then when you're sure that your code is correct, you can simply change n from 5 to 10,000.

- (b) Compute the mapping matrix T for each of the three methods (Jacobi, Gauss-Seidel, and SOR) via extracting the entries from A . Show your code.

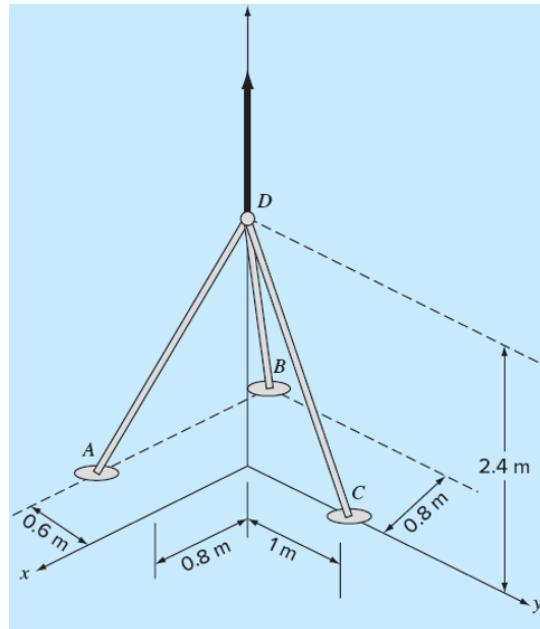


Figure 3: The uncomfortable, virtual forces acting on a tripod while an influencer takes pictures and performs obnoxious dance moves.

- (c) Compute the spectral radii for the transformation matrix T for the three methods. If any of the methods yield a $\rho(T) > 1$, find a way to generate a random matrix with a spectral radius smaller than one.
- (d) Compute the optimal ω^* for the SOR method.
- (e) Compute the solution of $Ax = b$ via $x^* = A^{-1}b$ or $x = A \ / \ b$ or $x = \text{mldivide}(A, b)$ on Matlab. Are there any differences?
- (f) Perform 100 iterations of the three methods [Jacobi, Gauss-Seidel, and SOR (with $\omega = 1.1$ and $\omega = \omega^*$)] and discuss the results. Include your code and showcase a plot of convergence. Compare the solution with $x^* = A^{-1}b$.
- (g) Is there a computational time difference between the three methods? Which one takes the longest to execute 100 iterations?