module 04
linear algebra for beginners

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scalars, vectors, matrices

- a scalar $x \in \mathbb{R}$, vector $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n \times m}$
- m is the number of columns, n defines number of rows (matrix is square if $n = m$, otherwise rectangular)
- identity matrix defined as I_n of size n

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- diagonal matrix is a matrix with only non-zero diagonal elements, zero elsewhere
- for matrix $A \in \mathbb{R}^{n \times n}$, a_{ij} defines the entry in the ith row and jth column
- a matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if $a_{ij} = a_{ji}$ for all i, j
- $A^\top$: matrix transpose, computed by switching the row and column indices

$$\left[A^\mathsf{T}\right]_{ij} = [A]_{ji}$$

- example: find $A^\top$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{bmatrix}$

  answer: $A^\top = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{bmatrix}$
- a matrix is symmetric if $A^\top = A$ (only applies to square matrices)

matrix operations

- addition: you cannot add/subtract two matrices/vectors of different dimensions
- that is, matrix addition $A + B$ only defined if A and B are same dimension $\mathbb{R}^{n\times m}$
- same thing for vectors $x \pm y$, both should be in $\mathbb{R}^n$
- vector-matrix multiplication Ax or xA only defined if dimensions commute
- when x multiplies A from the right, we say it's post-multiplication, otherwise pre
- you can never divide by a matrix (you can if you wanna send me to the ER)
- a scalar matrix multiplication cA where $c \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times m}$ means that c multiplies all entries in A, same thing for cx, scalar-vector multiplication
- cannot simply add $c + A$ unless you write $c\mathbf{1}_{n\times n} + A$, $\mathbf{1}_{n\times n}$ is square matrix of ones
- matrix-matrix multiplication AB only holds if A, B if $m_A = n_B$
- $\text{trace}(A)$: sum of the diagonal entries of A
- special matrices: lower triangular (L), upper triangular (U)
- what is a matrix-vector multiplication anyway?
- vector-vector multiplication xy or $x^\top y$ (dot product)

matrix-vector multiplications

- column version:

consider a matrix $A \in \mathbb{R}^{n\times m}$ with columns $a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$:

$$A = [a_1 \cdots a_m]$$

matrix-vector multiplication Ax, where $x \in \mathbb{R}^m$, can be interpreted as a linear combination of the columns of A, weighted by the entries of x:

$$Ax = x_1 a_1 + \ldots + x_m a_m$$

- row version:

consider a matrix $B \in \mathbb{R}^{p\times n}$ with rows b_i^T, $i = 1, \ldots, p$, where $b_i \in \mathbb{R}^n$:

$$B = \begin{bmatrix} b_1^T \\ \vdots \\ b_p^T \end{bmatrix}.$$

matrix-vector multiplication Bx, where $x \in \mathbb{R}^n$, can be interpreted as a column vector where each entry is the inner product between a row of B and x:

$$Bx = \begin{bmatrix} b_1^T x \\ \vdots \\ b_p^T x \end{bmatrix}.$$

interpretation of matrix-matrix multiplication: outer product

- consider a matrix $A \in \mathbb{R}^{m\times p}$ with columns $a_i \in \mathbb{R}^m$, $i = 1, \ldots, p$:

$$A = [a_1 \cdots a_p]$$

- consider a matrix $B \in \mathbb{R}^{p\times n}$ with rows b_i^T, $i = 1, \ldots, p$, where $b_i \in \mathbb{R}^n$:

$$B = \begin{bmatrix} b_1^T \\ \vdots \\ b_p^T \end{bmatrix}$$

- matrix-matrix product AB can be interpreted as a sum of matrices, where each matrix is the outer product between one column of A and one row of B:

$$AB = \sum_{i=1}^{p} a_i b_i^T$$

- example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 14 & 16 \\ 28 & 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

matrix determinants

- determinant defines a scaling factor of the linear transformation represented by a matrix A
- matrix determinant only defined for square matrices $A \in \mathbb{R}^{n\times n}$, notation $|A|$ or $\det(A)$
- examples:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - ge) = aei + bfg + cdh - ceg - bdi - afh$$

- matrix operations can change determinant values
- but transposing won't: $\det\left(A^\top\right) = \det(A)$
- another nice property: $\det(AB) = \det(A)\det(B)$
- another one

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)} = [\det(A)]^{-1}$$

eigenvalues and eigenvectors

- what are evalues/evectors? applications?
- evalues/vectors are only defined for square matrices
- for a matrix $A \in \mathbb{R}^{n\times n}$, we always have n evalues/evectors
  - some of these evalues might be distinct, real, repeated, imaginary
  - to find evalues(A), solve this equation (I_n: identity matrix of size n)

$$\det(\lambda I_n - A) = 0 \text{ or } \det(A - \lambda I_n) = 0 \Rightarrow a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

- eigenvectors: a number λ and a non-zero vector v satisfying

$$\boxed{Av = \lambda v \Rightarrow (A - \lambda I_n)v = 0}$$

  are called an eigenvalue and an eigenvector of A
- vectors $v_1, v_2, \ldots, v_k$ all in $\mathbb{R}^n$ are linearly (in)dependent if there exist (no) scalars $a_1, a_2, \ldots, a_k$ such that $\sum_{i=1}^k a_i v_i = 0_n$
- evectors $v_1, \ldots, v_n$ are linearly independent (they need to be)

example: calculating eigenvalues and eigenvectors

- find evalues and evectors for $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$
- step 1: solve $\det(A - \lambda I_2) = 0$

  $\det \begin{bmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = (5-\lambda)^2 - 4 = 0$

  $25 - 10\lambda + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 - 10\lambda + 21 = 0$

  $(\lambda - 7)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 7, \lambda_2 = 3$
- step 2: find eigenvector v_1 for $\lambda_1 = 7$

  solve $(A - 7I_2)v_1 = 0 \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  $-2x + 2y = 0 \Rightarrow x = y$. so we can choose $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- step 3: find eigenvector v_2 for $\lambda_2 = 3$

  solve $(A - 3I_2)v_2 = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  $2x + 2y = 0 \Rightarrow x = -y$. so we can choose $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

vector norms

norm of a vector $x \in \mathbb{R}^n$ is a function $\|.\| : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following three properties for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $a \in \mathbb{R}$.

1. *positive definiteness:* $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$
2. *scaling:* $\|ax\| = |a|\|x\|$
3. *triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$

- we use the notation $\|.\|$ for any norm satisfying the previous properties
- we use the notation $\|.\|_p$ for a specific norm, to be defined shortly
- the norm is also called the length of the vector

distance

the distance, or metric, between two points in $\mathbb{R}^n$ is defined as

$$d(x, y) = \|x - y\|.$$

the distance satisfies the following three properties.

1. *positive definiteness:* $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$
2. *symmetry:* $d(x, y) = d(y, x)$
3. *triangle inequality:* $d(x, z) \leq d(x, y) + d(y, z)$
4. or more generally: $||x + y|| \leq ||x|| + ||y||$

- special case of the triangular inequality: pythagorean theorem, given as follows
- if vectors x and y are orthogonal (i.e., $x^\top y = 0$) then $||x + y||^2 = ||x||^2 + ||y||^2$
- can compute angle between two vectors:

$$\cos(\theta) = \frac{x^\top y}{||x||||y||}$$

ℓ_p norms

for $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $p \geq 1$, the ℓ_p norm is defined as

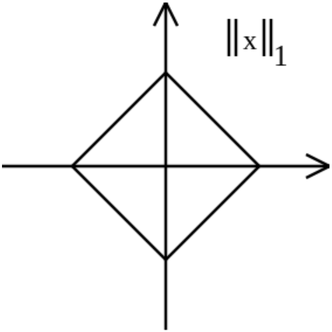
$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

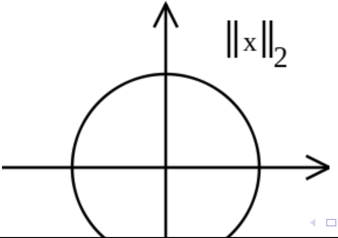
1. $p = 2$: euclidean norm $\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^n x_i^2}$
   - inner product $x^T x = [x_1 \quad \ldots \quad x_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots + x_n^2 = \|x\|_2^2$
2. $p = 1$: sum-abs-values $\|x\|_1 = \sum_i |x_i|$
3. $p = \infty$: max-abs-value $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$

example: calculating vector norms

- given a vector $x = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, let's compute different norms
- ℓ_1 norm (sum of absolute values):
  $\|x\|_1 = |-3| + |4| = 3 + 4 = 7$
- ℓ_2 norm (euclidean distance):
  $\|x\|_2 = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$
- ℓ_∞ norm (maximum absolute value):
  $\|x\|_\infty = \max(|-3|, |4|) = \max(3, 4) = 4$

norm examples





matrix norms

matrix norm is a function $\|\cdot\| : K^{m \times n} \to \mathbb{R}$ satisfying the following properties for all scalars α and matrices $A, B \in K^{m \times n}$

- $\|A\| \geq 0$ (positive-valued)
- $\|A\| = 0 \iff A = 0_{m,n}$ (definite)
- $\|\alpha A\| = |\alpha| \|A\|$ (absolutely homogeneous)
- $\|A + B\| \leq \|A\| + \|B\|$ (sub-additive or satisfying the triangle inequality)
- $\|AB\| \leq \|A\| \|B\|$ (sub-multiplicative property—super useful)

specific matrix norms

- frobenius-norm:

$$\|A\|_{\text{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\text{trace}\left(A^\top A\right)} = \sqrt{\sum_{i=1}^{\min\{m,n\}}\sigma_i^2(A)}$$

- 1-norm: $\|A\|_1 = \max\limits_{1\leq j\leq n}\sum\limits_{i=1}^{m}|a_{ij}|$: max absolute column sum of the matrix
- 2-norm: $\|A\|_2 = \sqrt{\lambda_{\max}\left(A^\top A\right)} = \sigma_{\max}(A)$ where $\sigma_{\max}(A)$ is the largest singular value of matrix A
  - note that $\|A\|_2 = \sigma_{\max}(A) \leq \|A\|_{\mathrm{F}} = \left(\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$
- infinity-norm: $\|A\|_\infty = \max\limits_{1\leq i\leq m}\sum\limits_{j=1}^{n}|a_{ij}|$: max absolute row sum of matrix
- max-norm: $\|A\|_{\max} = \max_{ij}|a_{ij}|$
- nuclear-norm: $\|A\|_* = \sum\limits_{i=1}^{\min\{m,n\}}\sigma_i(A)$

norm bounds and examples

for matrix $A \in \mathbb{R}^{m\times n}$ of rank r, the following inequalities hold

- $\|A\|_2 \leq \|A\|_F \leq \sqrt{r}\|A\|_2$
- $\|A\|_F \leq \|A\|_* \leq \sqrt{r}\|A\|_F$
- $\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn}\|A\|_{\max}$

example: $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 6 & 4 \\ 0 & 2 & 8 \end{bmatrix}$, then

- $\|A\|_1 = \max(|-3|+2+0; 5+6+2; 7+4+8) = \max(5, 13, 19) = 19$,
- $\|A\|_\infty = \max(|-3|+5+7; 2+6+4; 0+2+8) = \max(15, 12, 10) = 15$.
- $\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)} = \sqrt{\lambda_{\max}\left(\begin{bmatrix} 13 & 27 & 29 \\ 27 & 65 & 75 \\ 29 & 75 & 129 \end{bmatrix}\right)} \approx 13.686$

matrix inverse

- inverse of a generic 2by2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  – notice that $A^{-1}A = AA^{-1} = I_2$

- inverse of a generic 3by3 matrix:

$$A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{T} = \frac{1}{\det(A)} \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}$$

$$\begin{array}{ccc} A = (ei - fh) & D = -(bi - ch) & G = (bf - ce) \\ B = -(di - fg) & E = (ai - cg) & H = -(af - cd) \\ C = (dh - eg) & F = -(ah - bg) & I = (ae - bd) \end{array}$$

- $A \in \mathbb{R}^{n\times n}$ is invertible (nonsingular) if there is a matrix $B \in \mathbb{R}^{n\times n}$ such that $AB = I_n = BA$ and in this case, $B = A^{-1}$ is unique

some examples

- find the evalues/evectors and inverse of matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$
  - evalues: $\lambda_{1,2} = 5, -2$; evectors: $v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $v_2 = \begin{bmatrix} -\frac{4}{3} & 1 \end{bmatrix}^\top$
  - inverse computation:

$$\det(A) = (1)(2) - (4)(3) = 2 - 12 = -10$$

$$A^{-1} = \frac{1}{-10} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$$

- write A in the matrix diagonal transformation, i.e., $A = TDT^{-1}$ where D is the diagonal matrix containing the eigenvalues of A:

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1}$$

- only valid for matrices with distinct, real eigenvalues

matrix rank

- rank of a matrix: (A) is equal to the number of linearly independent rows or columns
  - example 1: $\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix}\right) = 1$ (row 2 is strictly a multiple of row 1)
  - example 2: $\left(\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}\right) =?$
- rank computation: reduce the matrix to a simpler form, generally row echelon form, by elementary row operations

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \to 2r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \to -3r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\to r_2 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to -2r_2 + r_1 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow (A) = 2$$

- for a matrix $A \in \mathbb{R}^{m \times n}$: $\text{rank}(A) \leq \min(m, n)$

matrix null space

- null space of any matrix A is the subspace $\mathcal{K}$ defined as follows:

$$\text{null}(A) = \ker(A) = \{x \in \mathcal{K} | Ax = \mathbf{0}\}$$

- null(A) has the following three properties:
  - null(A) always contains the zero vector, since $A0 = 0$
  - if $x \in \text{null}(A)$ and $y \in \text{null}(A)$, then $x + y \in \text{null}(A)$
  - if $x \in \text{null}(A)$ and c is a scalar, then $cx \in \text{null}(A)$
- example: compute null space of A

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ -4 & 2 & 3 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/16 & 0 \\ 0 & 1 & 13/8 & 0 \end{array}\right] \Rightarrow a = -\frac{1}{16}c, b = -\frac{13}{8}c \quad \Rightarrow \boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \begin{bmatrix} -1/16 \\ -13/8 \\ 1 \end{bmatrix} = \tilde{\alpha} \begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}}$$

ill-conditioning and condition numbers

- conditioning: a measure of how sensitive the output of a function $y = f(x) : \mathbb{R}^n \to \mathbb{R}$ to changes in the input argument $x \in \mathbb{R}^n$
- metric/measure for conditioning: condition function $\kappa(f(x))$
- ill-conditioned function: a function where tiny changes in x result in substantial changes in y
- well-conditioned function: opposite of ill-conditioned
- defining $\delta f(x) = f(x) - f(x + \delta x)$, where δx is a number smaller than ϵ
- absolute condition number:

$$\kappa_a(x) = \lim_{\varepsilon \to 0^+} \sup_{\|\delta x\| \leq \varepsilon} \frac{\|\delta f(x)\|}{\|\delta x\|} > 0$$

- relative condition number:

$$\kappa_r(x) = \lim_{\varepsilon \to 0^+} \sup_{\|\delta x\| \leq \varepsilon} \frac{\|\delta f(x)\| / \|f(x)\|}{\|\delta x\| / \|x\|} > 0$$

- $\kappa_r(x)$ is used more often, similar to why we use relative error and not absolute

explicit solution of a linear system of equations

with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, consider the system of equations

$$Ax = b$$

- this linear system of equations is one of the most solved problems
- special case: $m = n$, i.e., A is a square matrix
  - if A is invertible (or non-zero determinant, or not rank-deficient), then $x = A^{-1}b$ is the unique solution
  - if If A is not invertible, no unique solution
- general case $(m \neq n)$, we need to create new matrix inverse
  - that is, the moore-penrose pseudo inverse of A defined as $A^\dagger$
  - if A has linearly independent columns $(m > n)$ and hence $A^\top A$ is invertible, then

$$\textit{// left inverse:} \quad A^\dagger = \left(A^\top A\right)^{-1} A^\top \quad \Rightarrow \quad \boxed{x = A^\dagger b}$$

  - else if A has linearly independent rows $(m < n)$, then

$$\textit{// right inverse:} \quad A^\dagger = A^\top \left(AA^\top\right)^{-1} \quad \Rightarrow \quad \boxed{x = A^\dagger b}$$

linear systems of equations $Ax = b$

- this module is about solving $Ax = b$ where $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$
- so we have m equations, n unknowns: sometimes $m >> n$ (overdetermined) or $n >> m$ (underdetermined) or $n = m$
- example: find a parabola that goes through the points $(x, y) = \{(1,1), (2,2), (3,5)\}$
- we first focus on systems of n equations n unknowns $Ax = b$
- this can be solved via writing $x = A^{-1}b$ but inverse computation is difficult

matrix inverse theorem

for a matrix $A \in \mathbb{R}^{n \times n}$, the following are all equivalent:

1. A is an invertible matrix
2. columns of A are linearly independent
3. $Ax = 0$ has only the trivial solution $x = 0$
4. $Ax = b$ has unique solution for each $b \in \mathbb{R}^n$
5. the number 0 is not an evalue of A
6. $\det(A) \neq 0$
7. $A^\top$ is invertible and $(A^\top)^{-1} = (A^{-1})^\top$

example: setting up $Ax = b$

- let's solve the parabola example: $y = c_1x^2 + c_2x + c_3$ through $(1,1),(2,2),(3,5)$
- plug in each point to construct our system of equations:
- for $(x, y) = (1, 1)$:
  $c_1(1)^2 + c_2(1) + c_3 = 1 \Rightarrow c_1 + c_2 + c_3 = 1$
- for $(x, y) = (2, 2)$:
  $c_1(2)^2 + c_2(2) + c_3 = 2 \Rightarrow 4c_1 + 2c_2 + c_3 = 2$
- for $(x, y) = (3, 5)$:
  $c_1(3)^2 + c_2(3) + c_3 = 5 \Rightarrow 9c_1 + 3c_2 + c_3 = 5$
- write it in matrix form $Ac = b$:
  $$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$
- this is a 3×3 system with a unique solution because A is invertible

outline of methods to solve $Ax = b$

many, many methods to solve $Ax = b$

- graphical methods
- finite-step methods (tolerance-free, exact, terminate after fixed number of steps) for small n:
    - cramer's rule
    - method of elimination
    - naive gauss
    - gauss-jordan
    - lu decomposition
    - ...
- iterative methods with tolerance-defined stopping criteria, works well for large n

module summary

- brief review on basics in linear algebra
- need to be comfortable handling matrices: multiplication, rank, determinant, evalues, etc..
- importance of assessing condition number of matrices (datasets) and functions
- solving linear systems of equations (over/under-determined)
- good engineering = linear algebra + calculus + engineering domain expertise + programming
- need to be good at working with matrices