

1. Functions, sets, and basic function types.

Solution

For

$$f(x_1, x_2) = \sqrt{9 - x_1^2 - x_2^2},$$

the quantity under the square root must be nonnegative. Therefore

$$9 - x_1^2 - x_2^2 \geq 0 \iff x_1^2 + x_2^2 \leq 9.$$

Hence

$$\text{dom}(f) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 9\}.$$

The minimum value of  $f$  is 0, attained on the circle  $x_1^2 + x_2^2 = 9$ , and the maximum is 3, attained at the origin. Thus

$$\text{range}(f) = [0, 3].$$

For

$$S = \{(x_1, 0) : -2 \leq x_1 \leq 2\},$$

we have

$$f(x_1, 0) = \sqrt{9 - x_1^2}.$$

Over  $[-2, 2]$ , its maximum is 3 at  $x_1 = 0$  and its minimum is  $\sqrt{5}$  at  $x_1 = \pm 2$ . Therefore

$$f(S) = [\sqrt{5}, 3].$$

For  $T = [2, 3]$ , the inverse image is determined by

$$2 \leq \sqrt{9 - x_1^2 - x_2^2} \leq 3.$$

Squaring gives

$$4 \leq 9 - x_1^2 - x_2^2 \leq 9,$$

so

$$0 \leq x_1^2 + x_2^2 \leq 5.$$

Thus

$$f^{-1}(T) = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 5\},$$

the closed disk of radius  $\sqrt{5}$  centered at the origin.

Finally,

$$g(x) = 3x_1 - 2x_2$$

is linear,

$$h(x) = 3x_1 - 2x_2 + 5$$

is affine, and

$$q(x) = \frac{1}{2}x^T \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} -2 \\ 3 \end{bmatrix}^T x + 1$$

is quadratic.

## 2. Gradient, Hessian, and Taylor approximation.

| Solution                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
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| <p>The function is</p> $f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2 - 4x_1 + 2x_2 + 7.$ <p>Its first derivatives are</p> $\frac{\partial f}{\partial x_1} = 2x_1 + 2x_2 - 4, \quad \frac{\partial f}{\partial x_2} = 2x_1 + 6x_2 + 2.$ <p>Therefore</p> $\nabla f(x) = \begin{bmatrix} 2x_1 + 2x_2 - 4 \\ 2x_1 + 6x_2 + 2 \end{bmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}.$ <p>At</p> $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$ <p>we obtain</p> $f(x_0) = 3, \quad \nabla f(x_0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}, \quad \nabla^2 f(x_0) = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}.$ <p>The first-order approximation is</p> $f_{\text{lin}}(x) = f(x_0) + \nabla f(x_0)^T (x - x_0),$ <p>so</p> $f_{\text{lin}}(x) = 3 - 4(x_1 - 1) - 2(x_2 + 1) = 5 - 4x_1 - 2x_2.$ <p>The second-order approximation is</p> $f_{\text{quad}}(x) = 3 + \begin{bmatrix} -4 & -2 \end{bmatrix} (x - x_0) + \frac{1}{2} (x - x_0)^T \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} (x - x_0).$ <p>Because <math>f</math> itself is a quadratic function with constant Hessian, the second-order Taylor formula reproduces <math>f</math> exactly.</p> <p>At <math>x = [1.1 \ -0.8]^T</math>,</p> $f(x) = 2.37, \quad f_{\text{lin}}(x) = 2.20, \quad f_{\text{quad}}(x) = 2.37.$ <p>The second-order approximation is exact for every <math>x</math>, not only at this evaluation point.</p> |

## 3. Vector norms, distance, and inequalities.

| Solution                                                                                                                                                                                                                                                                                                                                                                   |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>For</p> $x = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix},$ <p>the norms are</p> $\ x\ _1 =  -2  +  1  +  3  = 6,$ $\ x\ _2 = \sqrt{4 + 1 + 9} = \sqrt{14} \approx 3.7417, \quad \ x\ _\infty = 3,$ <p>and</p> $\ y\ _1 = 7, \quad \ y\ _2 = \sqrt{1 + 16 + 4} = \sqrt{21} \approx 4.5826, \quad \ y\ _\infty = 4.$ |

Since

$$x - y = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix},$$

the three distances are

$$d_1(x, y) = 9, \quad d_2(x, y) = \sqrt{35} \approx 5.9161, \quad d_\infty(x, y) = 5.$$

For Cauchy–Schwarz,

$$x^T y = (-2)(1) + (1)(-4) + (3)(2) = 0.$$

Hence

$$|x^T y| = 0 \leq \sqrt{14}\sqrt{21} = \sqrt{294} \approx 17.1464.$$

For Hölder with  $p = 1$  and  $q = \infty$ ,

$$\sum_{i=1}^3 |x_i y_i| = 2 + 4 + 6 = 12,$$

while

$$\|x\|_1 \|y\|_\infty = (6)(4) = 24.$$

Thus  $12 \leq 24$ .

For

$$N(z) = \sqrt{z_1^2 + 4z_2^2},$$

define

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$N(z) = \|Dz\|_2.$$

Since  $D$  is nonsingular and the Euclidean norm is a norm,  $N$  satisfies positive definiteness, absolute homogeneity, and the triangle inequality.

On the other hand,

$$G(z) = \|z\|_2^2$$

is not a norm because absolute homogeneity fails. In general,

$$G(az) = a^2 G(z),$$

whereas a norm must satisfy

$$G(az) = |a| G(z).$$

For example,  $G(2z) = 4G(z)$  rather than  $2G(z)$ .

#### 4. Matrix norms, singular values, and norm bounds.

**Solution**

For

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix},$$

the maximum absolute column sum is

$$\|A\|_1 = \max\{1+2, 2+2\} = 4,$$

and the maximum absolute row sum is

$$\|A\|_\infty = \max\{1+2, 2+2\} = 4.$$

The Frobenius norm is

$$\|A\|_F = \sqrt{1^2 + (-2)^2 + 2^2 + 2^2} = \sqrt{13} \approx 3.6056.$$

Next,

$$A^T A = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}.$$

The characteristic polynomial is

$$\det\left(\begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} - \lambda I\right) = (5 - \lambda)(8 - \lambda) - 4 = \lambda^2 - 13\lambda + 36.$$

Therefore

$$\lambda_1 = 9, \quad \lambda_2 = 4.$$

The singular values are defined as the square roots of these eigenvalues:

$$\sigma_1 = 3, \quad \sigma_2 = 2.$$

Hence

$$\|A\|_2 = 3, \quad \|A\|_* = 3 + 2 = 5.$$

Since  $\det(A) = 6 \neq 0$ ,  $\text{rank}(A) = 2$ . The stated bound becomes

$$3 \leq \sqrt{13} \leq 3\sqrt{2},$$

which is true numerically:

$$3 \leq 3.6056 \leq 4.2426.$$

For

$$B = \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix},$$

we obtain

$$AB = \begin{bmatrix} -7 & 1 \\ 4 & 2 \end{bmatrix}.$$

Thus

$$\|AB\|_F = \sqrt{49 + 1 + 16 + 4} = \sqrt{70} \approx 8.3666.$$

Also

$$\|B\|_F = \sqrt{11},$$

so

$$\|A\|_F \|B\|_F = \sqrt{13}\sqrt{11} = \sqrt{143} \approx 11.9583.$$

Therefore

$$\|AB\|_F \leq \|A\|_F \|B\|_F.$$

Finally,  $\|A\|_F$  is exactly the Euclidean norm of the vector obtained by stacking all entries of  $A$ . This immediately gives nonnegativity, definiteness, absolute homogeneity, and the triangle inequality. The Frobenius norm also satisfies the matrix submultiplicative inequality.

5. Eigenvalues, eigenvectors, inverse, and diagonalization.

Solution

Let

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{bmatrix} = (4 - \lambda)(3 - \lambda) - 2.$$

Thus

$$\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2),$$

so

$$\lambda_1 = 5, \quad \lambda_2 = 2.$$

For  $\lambda_1 = 5$ ,

$$(A - 5I)v = 0 \implies \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} v = 0,$$

which gives  $v_2 = v_1$ . One eigenvector is

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For  $\lambda_2 = 2$ ,

$$(A - 2I)v = 0 \implies \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} v = 0,$$

which gives  $v_2 = -2v_1$ . One eigenvector is

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The determinant of  $A$  is

$$\det(A) = 12 - 2 = 10,$$

so

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}.$$

Direct multiplication gives

$$AA^{-1} = I_2.$$

Using the eigenvectors as columns,

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since

$$T^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix},$$

we have

$$A = TDT^{-1}.$$

A numerical eigenvalue calculation returns the same eigenvalues and eigenvector directions, up to arbitrary scaling and sign of the eigenvectors.

6. Square-image reconstruction from eigenanalysis of  $A^T A$ .

Solution

The supplied bridge image is represented by

$$A \in \mathbb{R}^{128 \times 128},$$

with pixel intensities scaled to  $[0, 1]$ . The default numerical rank is

$$\text{rank}(A) = 128.$$

Thus the stored image matrix is technically full rank, even though its visually important content can be represented well by far fewer modes.

Form

$$C = A^T A.$$

For any  $z \in \mathbb{R}^{128}$ ,

$$z^T C z = z^T A^T A z = (A z)^T (A z) = \|A z\|_2^2 \geq 0.$$

Therefore  $C$  is positive semidefinite and all of its exact eigenvalues are nonnegative. Tiny negative values can only arise from floating-point roundoff; clipping such values to zero before taking a square root is appropriate.

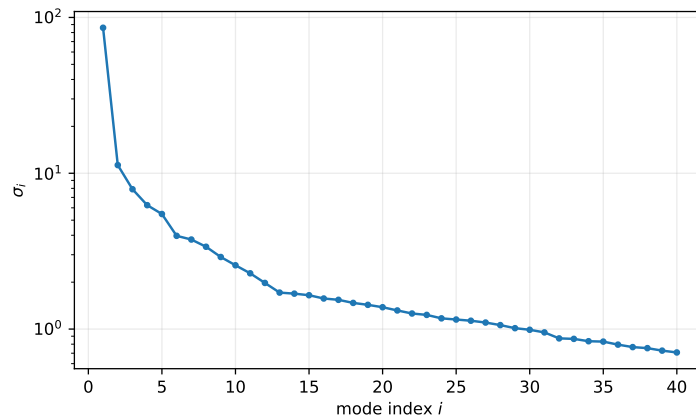
The first ten singular values obtained from

$$\sigma_i = \sqrt{\lambda_i(C)}$$

are approximately

| $i$        | 1       | 2       | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |
|------------|---------|---------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\sigma_i$ | 85.8746 | 11.2816 | 7.8992 | 6.2405 | 5.4759 | 3.9669 | 3.7554 | 3.3754 | 2.9009 | 2.5673 |

The rapid decrease is visible in the following plot.



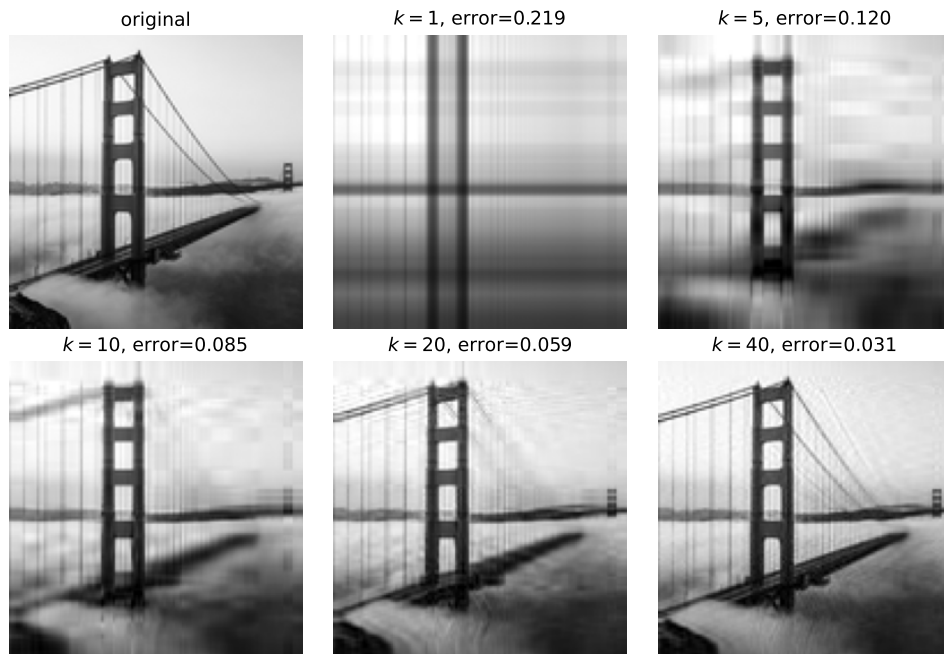
For each nonzero  $\sigma_i$ , define

$$u_i = \frac{A v_i}{\sigma_i}.$$

For a chosen  $k$ ,

$$A_k = U_k \Sigma_k V_k^T = \sum_{i=1}^k \sigma_i u_i v_i^T.$$

The reconstructions are shown below.



The relative Frobenius errors are

| $k$                       | 1      | 5      | 10     | 20     | 40     |
|---------------------------|--------|--------|--------|--------|--------|
| $\ A - A_k\ _F / \ A\ _F$ | 0.2188 | 0.1204 | 0.0850 | 0.0595 | 0.0313 |

A rank-20 reconstruction is already visually reasonable for the major bridge structure, while  $k = 40$  preserves substantially more fine detail.

The eigenvectors of  $A^T A$  identify linear combinations of image columns that capture the strongest structured variation. The associated eigenvalues quantify the squared strength of those directions. When the eigenvalues decay rapidly, most image energy is concentrated in a relatively small number of directions. Retaining only those directions gives a compact approximation with small reconstruction error.

## 7. Rank, determinant, and null space.

### Solution

Consider

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Perform row operations:

$$R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 - R_1.$$

This gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

There are two nonzero rows, so

$$\text{rank}(M) = 2.$$

To find the null space, solve  $Mx = 0$ . The independent equations are

$$x_1 + 2x_2 + x_3 = 0, \quad x_1 + x_2 = 0.$$

The second equation gives  $x_1 = -x_2$ . Substitution into the first gives  $x_3 = -x_2$ . Letting  $x_2 = -\alpha$  gives

$$x = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Hence

$$\text{Null}(M) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Because the second row is twice the first row,

$$\det(M) = 0.$$

The conclusions are consistent:  $M$  is  $3 \times 3$  but has rank 2, so it is not invertible, has a non-trivial null space, and has at least one zero eigenvalue.

## 8. Symmetry and definiteness.

### Solution

For

$$A_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

the leading principal minors are

$$\Delta_1 = 2, \quad \Delta_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3,$$

and

$$\Delta_3 = \det(A_1) = 4.$$

All are positive, so Sylvester's criterion gives

$$A_1 \succ 0.$$

Thus  $A_1$  is positive definite.

For

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

the one-by-one principal minors are both 1 and

$$\det(A_2) = 0.$$

All principal minors are nonnegative, so

$$A_2 \succeq 0.$$

It is not positive definite because its determinant is zero; equivalently, its eigenvalues are 2 and 0.

For

$$A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\det(A_3) = 1 - 4 = -3 < 0.$$

The eigenvalues therefore have opposite signs; in fact they are 3 and  $-1$ . Hence  $A_3$  is indefinite.

Finally,

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Its leading principal minors are all zero, but consider the principal submatrix formed by rows and columns 2 and 3:

$$C_{\{2,3\}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Its determinant is

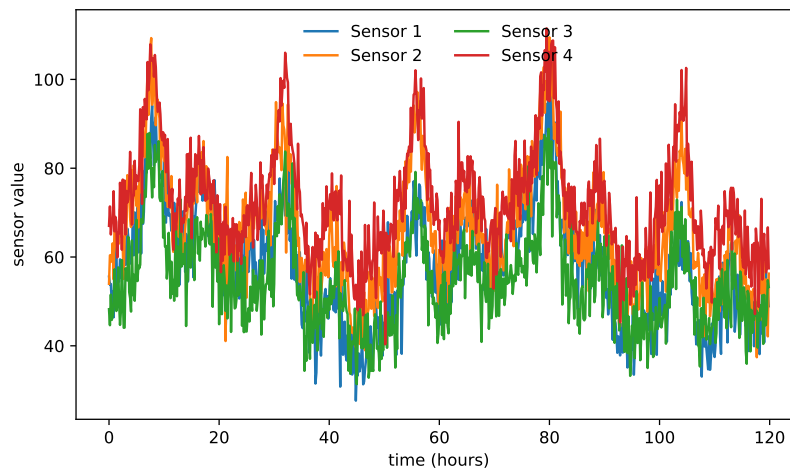
$$1 - 4 = -3 < 0.$$

Therefore  $C$  is not positive semidefinite. The example shows why positive semidefiniteness requires all principal minors to be nonnegative; checking only the nested sequence of leading principal minors can miss a negative direction in another coordinate subset.

## 9. Low-rank structure in noisy time-series data.

### Solution

The first four sensor signals are shown below. The raw data contain a strong common temporal component together with sensor-specific noise.



Let

$$X \in \mathbb{R}^{16 \times 720}$$

contain one sensor per row and let  $X_c$  denote the row-centered matrix. The default numerical rank is

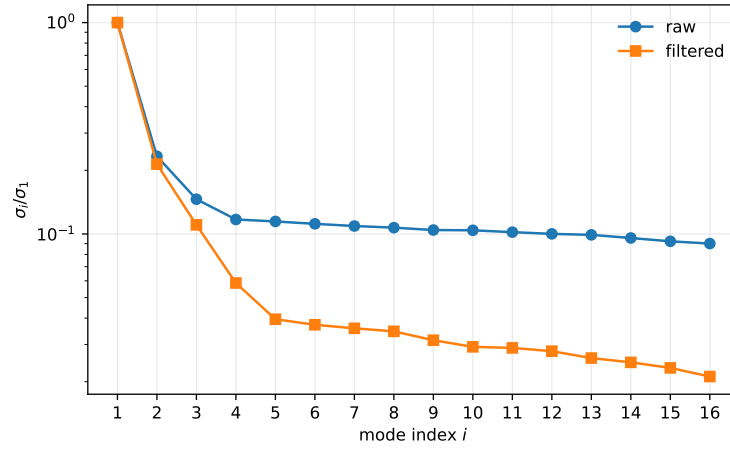
$$\text{rank}(X_c) = 16.$$

After applying the 11-sample moving average to every row and centering again,

$$\text{rank}(X_f) = 16.$$

Thus filtering does not generally make the matrix exactly rank deficient. Small independent components remain numerically nonzero.

The singular values are obtained from eigenvalues. Because  $X_c X_c^T$  is only  $16 \times 16$ , it is computationally convenient to use its eigenvalues; its positive eigenvalues are the same as the positive eigenvalues of  $X_c^T X_c$ . The normalized spectra are shown below.



The raw and filtered singular values begin approximately as

|          | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ |
|----------|------------|------------|------------|------------|------------|
| raw      | 1327.86    | 309.54     | 193.85     | 155.49     | 152.15     |
| filtered | 1287.83    | 275.57     | 142.20     | 75.48      | 50.86      |

The filtered spectrum decays much more sharply after the leading modes.

Define

$$E(r) = \frac{\sum_{i=1}^r \sigma_i^2}{\sum_i \sigma_i^2}.$$

For the raw centered data, the first 14 modes account for about 98.63% of the energy and the first 15 account for about 99.33%. Therefore the 99% effective rank is

$$r_{99,\text{raw}} = 15.$$

For the filtered data, the first 4 modes account for about 98.96% and the first 5 account for about 99.11%. Therefore

$$r_{99,\text{filtered}} = 5.$$

Using the same eigenvector-based reconstruction used in Problem 6, the rank-5 approximation of the centered filtered matrix has relative Frobenius error

$$\frac{\|X_f - X_{f,5}\|_F}{\|X_f\|_F} \approx 0.0945.$$

The important distinction is between exact numerical rank and effective rank. Both raw and filtered matrices remain full row rank, but filtering removes much of the small, uncorrelated noise. The remaining correlated time-series structure is then concentrated in only a few dominant directions, reducing the effective rank from 15 to 5.

10. Moore–Penrose inverse for rectangular systems.

Solution

For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

the two columns are linearly independent. Compute

$$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

so

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Therefore

$$A^\dagger = (A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Then

$$x = A^\dagger b = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Indeed,

$$Ax = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = b.$$

For

$$G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

the two rows are linearly independent. We have

$$GG^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (GG^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Hence

$$G^\dagger = G^T (GG^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$z = G^\dagger c = \frac{1}{3} \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 4/3 \end{bmatrix}.$$

Verification gives

$$Gz = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c.$$

The solution is not unique because  $G$  has three unknowns but rank 2. Its null space is one-dimensional:

$$\text{Null}(G) = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Hence every solution can be written as

$$z = \begin{bmatrix} 5/3 \\ -1/3 \\ 4/3 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Equivalently, setting  $z_3 = t$  gives  $z_1 = 3 - t$  and  $z_2 = 1 - t$ .

11. A small traffic network: write and solve  $Aq = b$  manually.

#### Solution

Flow conservation gives one equation at each of nodes 2 through 5. At node 2,

$$q_1 = q_2 + q_3 + 80,$$

so

$$q_1 - q_2 - q_3 = 80.$$

At node 3,

$$q_2 = 220.$$

At node 4,

$$q_3 = q_4 + 60,$$

so

$$q_3 - q_4 = 60.$$

At node 5,

$$q_4 = 140.$$

Therefore

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_A \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 80 \\ 220 \\ 60 \\ 140 \end{bmatrix}}_b.$$

Back substitution gives

$$q_4 = 140, \quad q_3 = 200, \quad q_2 = 220, \quad q_1 = 500.$$

Thus

$$q = \begin{bmatrix} 500 \\ 220 \\ 200 \\ 140 \end{bmatrix} \text{ vehicles/hour.}$$

The coefficient matrix is upper triangular with diagonal entries all equal to 1, so

$$\det(A) = 1, \quad \text{rank}(A) = 4.$$

Therefore  $A$  is invertible and the flow solution is unique.

12. A larger traffic network: incidence matrices and coding.

## Solution

The supplied incidence matrix and injection vector have dimensions

$$B \in \mathbb{R}^{14 \times 20}, \quad s \in \mathbb{R}^{14}.$$

The computed rank is

$$\text{rank}(B) = 13,$$

and

$$\sum_{i=1}^{14} s_i = 0.$$

Every incidence-matrix column contains one +1 and one -1, so

$$\mathbf{1}^T B = 0.$$

Consequently, the sum of all nodal conservation equations is identically zero, making one nodal equation redundant.

After removing row 1,

$$B_r \in \mathbb{R}^{13 \times 20}, \quad \text{rank}(B_r) = 13.$$

There are 20 unknown flows and only 13 independent conservation equations, leaving

$$20 - 13 = 7$$

undetermined degrees of freedom. Conservation alone therefore does not determine a unique link-flow vector.

The right-pseudoinverse solution

$$q_0 = B_r^T (B_r B_r^T)^{-1} s_r$$

is

| edge | $q_{0,i}$ | edge | $q_{0,i}$ |
|------|-----------|------|-----------|
| 1    | 613.264   | 11   | 168.978   |
| 2    | 586.736   | 12   | 62.089    |
| 3    | 247.226   | 13   | 100.000   |
| 4    | 170.744   | 14   | 185.293   |
| 5    | 211.821   | 15   | 197.272   |
| 6    | 252.644   | 16   | 66.759    |
| 7    | 121.022   | 17   | 104.775   |
| 8    | 59.445    | 18   | 102.911   |
| 9    | 143.241   | 19   | 80.555    |
| 10   | 90.225    | 20   | -13.978   |

with

$$\|B_r q_0 - s_r\|_2 \approx 7.0 \times 10^{-13}.$$

The negative value on edge 20 is not an inconsistency. It indicates that this minimum-norm algebraic solution sends that edge's flow opposite to its chosen reference orientation. Conservation alone permits many other solutions.

The measurements are on edges 14 through 20. Therefore the selector matrix has the particularly simple form

$$H = \begin{bmatrix} 0_{7 \times 13} & I_7 \end{bmatrix}.$$

The augmented system is

$$A_{\text{aug}} = \begin{bmatrix} B_r \\ H \end{bmatrix} \in \mathbb{R}^{20 \times 20}.$$

Its rank is

$$\text{rank}(A_{\text{aug}}) = 20,$$

so the augmented system has a unique solution. The recovered flows are

| edge | $q_i$ | edge | $q_i$ |
|------|-------|------|-------|
| 1    | 650   | 11   | 140   |
| 2    | 550   | 12   | 130   |
| 3    | 300   | 13   | 100   |
| 4    | 280   | 14   | 60    |
| 5    | 320   | 15   | 45    |
| 6    | 260   | 16   | 30    |
| 7    | 150   | 17   | 25    |
| 8    | 120   | 18   | 35    |
| 9    | 180   | 19   | 20    |
| 10   | 170   | 20   | 15    |

in vehicles/hour. The residual is zero to numerical precision:

$$\|A_{\text{aug}}q - b_{\text{aug}}\|_2 = 0$$

for the supplied data.

The conservation equations determine only net balance at nodes. In a meshed network they cannot distinguish how flow circulates among alternate paths. The seven independent link measurements remove exactly the seven remaining degrees of freedom and make the complete flow vector identifiable.

### 13. Writing a multi-input, multi-output infrastructure model in state-space form.

#### Solution

Define

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}.$$

Collecting coefficients from the four differential equations gives

$$A = \begin{bmatrix} -0.30 & 0.10 & 0.04 & 0 \\ 0.08 & -0.25 & 0 & 0.07 \\ 0.05 & 0 & -0.20 & 0.06 \\ 0 & 0.10 & 0.08 & -0.22 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.50 & 0 \\ 0.20 & 0 \\ 0 & 0.40 \\ 0 & 0.15 \end{bmatrix}.$$

From

$$y_1 = x_1, \quad y_2 = x_2 + x_4, \quad y_3 = x_3 + 0.05u_2,$$

we obtain

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.05 \end{bmatrix}.$$

Therefore

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

The dimensions are

$$A \in \mathbb{R}^{4 \times 4}, \quad B \in \mathbb{R}^{4 \times 2}, \quad C \in \mathbb{R}^{3 \times 4}, \quad D \in \mathbb{R}^{3 \times 2},$$

consistent with 4 states, 2 inputs, and 3 outputs.

#### 14. Numerically solving the state-space model with ode45 or solve\_ivp.

##### Solution

The numerical model is

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with

$$x(0) = \begin{bmatrix} 2.0 \\ 1.5 \\ 1.0 \\ 0.5 \end{bmatrix}$$

and piecewise-constant input

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

A solver function therefore returns

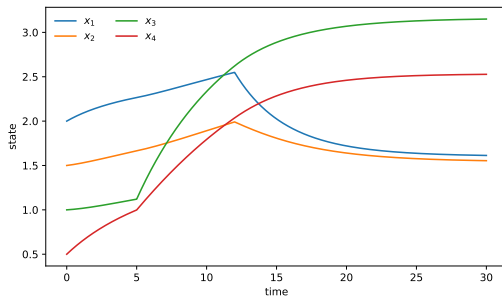
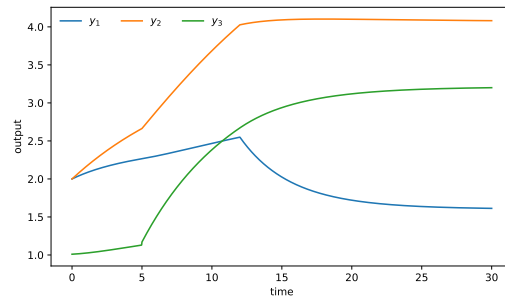
$$f(t, x) = Ax + Bu(t).$$

The two input discontinuities occur at  $t = 5$  and  $t = 12$ . A robust implementation may either allow the adaptive solver to step through these changes with a sufficiently small maximum step or integrate separately over the three intervals  $[0, 5]$ ,  $[5, 12]$ , and  $[12, 30]$ .

The simulated states and the outputs computed from

$$y(t) = Cx(t) + Du(t)$$

are shown below.



At  $t = 30$ ,

$$x(30) \approx \begin{bmatrix} 1.61308 \\ 1.55404 \\ 3.15011 \\ 2.52733 \end{bmatrix}.$$

Since  $u_2(30) = 1$ , the final output is

$$y(30) = Cx(30) + Du(30) \approx \begin{bmatrix} 1.61308 \\ 4.08136 \\ 3.20011 \end{bmatrix}.$$

The first output equals  $x_1$ . The second is  $x_2 + x_4$ , giving

$$1.55404 + 2.52733 = 4.08137$$

up to rounding. The third equals  $x_3 + 0.05u_2$ , so

$$3.15011 + 0.05 = 3.20011.$$

The state vector contains the internal variables required to evolve the model in time. The input vector represents externally specified actuation or forcing. The output vector contains the quantities selected for measurement or observation and may depend directly on both the current state and the input.