

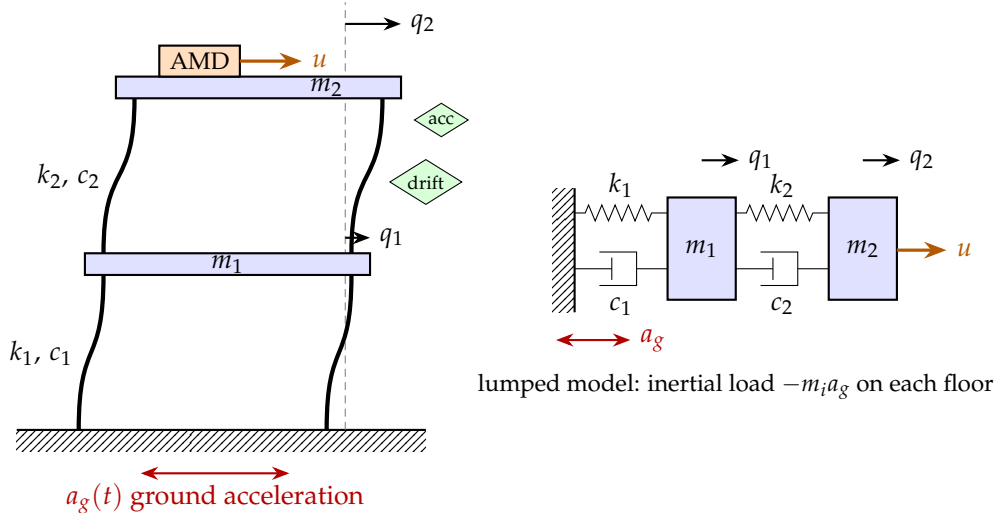
Due date of the homework: **Monday, October 26th, midnight.** Submit one PDF together with the source and code files. Show the mathematical work, label every plot (axes, units, legend), and check numerical answers against your hand calculations. You may use MATLAB or Python. Up to you.

Problem selection. There are 18 problems. Solve **any 13 of them.** **Problem 18 (MPC) is mandatory,** so you choose 12 of Problems 1–17 and add Problem 18. When drafting your solutions, if you plan to skip Problem x , just say you skipped it instead of changing the numbering of the problems. Also, each problem refers to the relevant slides of the Module 4 deck so you can read them before you start...or you have been so observant during the class, in that case you can start right away. .

A. State space modeling

1. **Two story shear building with an active mass driver.** [Review Slides 8/51, 9/51, and 11/51 in Module 4.pdf, uploaded to Brightspace]

The frame below is idealized as a shear building. Floor i has mass m_i ; the columns of story i act as a lateral spring of stiffness k_i and a viscous damper c_i . Let q_1 and q_2 be the floor displacements *relative to the ground*, and let the ground accelerate by $a_g(t)$ during an earthquake. An active mass driver (AMD) on the roof applies a horizontal control force $u(t)$ to floor 2. Two sensors are installed: a drift sensor that reads the second story drift $q_2 - q_1$, and a roof accelerometer that reads the *absolute* acceleration $\ddot{q}_2 + a_g$.



- (a) Follow the modeling workflow of Slide 9/51: identify the stored quantities, the states, the chosen input, the disturbance, and the measured outputs. Explain why a_g is a disturbance and not an input.
- (b) In the relative frame, the ground acceleration acts on floor i as the inertial force $-m_i a_g$. Derive the two equations of motion and write the model

$$\dot{x} = Ax + Bu + Ea_g, \quad y = Cx + Du, \quad x = [q_1 \quad q_2 \quad \dot{q}_1 \quad \dot{q}_2]^T,$$

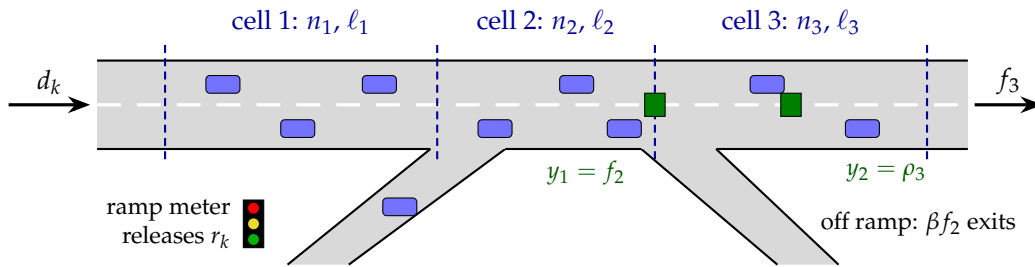
giving A, B, E, C, D symbolically.

- (c) Evaluate the matrices for $m_1 = m_2 = 1.0 \times 10^5$ kg, $k_1 = k_2 = 4.0 \times 10^7$ N/m, and $c_1 = c_2 = 1.0 \times 10^5$ N·s/m. Check the units of every row.

- (d) Compute the eigenvalues of A . Writing each complex pair as $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, report both natural frequencies (rad/s and Hz) and damping ratios. Is the uncontrolled building stable?
- (e) Explain why the accelerometer output needs $D \neq 0$ while the drift output does not. Does a_g appear directly in the accelerometer equation?
- (f) Suppose the AMD is replaced by an active brace that pushes floors 1 and 2 apart with equal and opposite forces $\pm u$. Which matrix changes, and what is its new value?

2. **Freeway corridor with ramp metering.** [Review Slides 6/51, 7/51, 9/51, and 13/51 in Module 4.pdf, uploaded to Brightspace]

A freeway segment is split into three cells of lengths ℓ_1, ℓ_2, ℓ_3 . Let $n_i(k)$ be the number of vehicles in cell i at time kT_s . In free flow, vehicles leave cell i at the rate $f_i = v_f n_i / \ell_i$ (veh/h), where v_f is the free flow speed. The upstream mainline demand d_k (veh/h) enters cell 1 and is not chosen. A ramp meter releases r_k (veh/h) into cell 2; this is the control input. A fraction β of the flow leaving cell 2 exits at an off ramp before reaching cell 3. Loop detectors measure the flow f_2 leaving cell 2 and the density $\rho_3 = n_3 / \ell_3$ of cell 3.



- (a) Write “vehicles in minus vehicles out over one sample” for each cell, and obtain

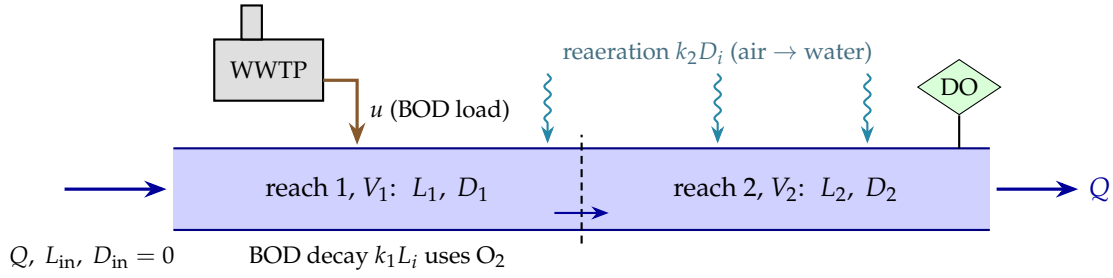
$$x_{k+1} = A_d x_k + B_d r_k + E_d d_k, \quad y_k = C x_k, \quad x_k = [n_1(k) \quad n_2(k) \quad n_3(k)]^T,$$

with all matrices written symbolically in terms of T_s, v_f, ℓ_i, β .

- (b) Evaluate the matrices for $v_f = 90$ km/h, $\ell = (0.5, 0.4, 0.5)$ km, $\beta = 0.2$, and $T_s = 10$ s. Be careful with units: flows are in veh/h.
- (c) Each diagonal entry of A_d equals $1 - T_s v_f / \ell_i$. Explain physically why $T_s v_f / \ell_i \leq 1$ must hold, and what goes wrong otherwise. Which of the four model classes on Slide 7/51 does this model belong to?
- (d) For constant $d = 4000$ veh/h and $r = 600$ veh/h, compute the equilibrium vehicle counts, the densities in veh/km, and the detector readings.
- (e) The linear free flow model is valid only while every cell is below the critical density $\rho_c = 60$ veh/km. With $d = 4000$ veh/h, find the largest constant ramp flow r that keeps cell 2 at or below ρ_c . What physical behavior would the linear model miss if the meter released more?
- (f) Find the eigenvalues of A_d . Is the model stable? Explain why A_d is lower triangular.

3. **Oxygen sag in a river below a wastewater treatment plant.** [Review Slides 8/51, 9/51, 12/51, and 13/51 in Module 4.pdf, uploaded to Brightspace]

A river is modeled as two well mixed reaches in series. The river flow is Q and the reach volumes are V_1 and $V_2 = 1.25 V_1$. In each reach, L_i is the biochemical oxygen demand (BOD, mg/L) and $D_i = \text{DO}_{\text{sat}} - \text{DO}_i$ is the dissolved oxygen deficit (mg/L). BOD decays at the rate $k_1 L_i$, consuming oxygen and raising the deficit. Reaeration from the atmosphere lowers the deficit at the rate $k_2 D_i$ (Streeter and Phelps kinetics). A treatment plant discharges into reach 1 with BOD loading $u = W / V_1$ (mg/L/day), the regulated input. The upstream water carries BOD L_{in} (a disturbance) and is saturated with oxygen ($D_{\text{in}} = 0$). A DO sensor sits at the downstream end of reach 2.



Use $a_1 = Q/V_1 = 0.5 \text{ day}^{-1}$, $a_2 = Q/V_2 = 0.4 \text{ day}^{-1}$, $k_1 = 0.3 \text{ day}^{-1}$, $k_2 = 0.6 \text{ day}^{-1}$, and $DO_{\text{sat}} = 9 \text{ mg/L}$.

- Identify the stored quantities, states, input, disturbance, and output.
- Write a mass balance for BOD in each reach. Starting from the oxygen balance

$$\dot{D}_i = a_i(DO_{\text{upstream}} - DO_i) - k_1 L_i + k_2(DO_{\text{sat}} - DO_i),$$

derive the deficit equations for $\dot{D}_1$ and $\dot{D}_2$.

- Write the model $\dot{x} = Ax + Bu + EL_{in}$ with $x = [L_1 \ D_1 \ L_2 \ D_2]^T$, numerically. The measured DO, $y = DO_{\text{sat}} - D_2$, is affine rather than linear. Rewrite the output so that the model is linear.
- For $L_{in} = 2 \text{ mg/L}$ and $u = 6 \text{ mg/L/day}$, compute the steady state and the DO concentration in reach 2.
- The water quality standard requires $DO_2 \geq 5 \text{ mg/L}$. Find the largest permissible constant loading u , and plot the steady DO_2 as a function of u .
- Find the eigenvalues of A and decide stability. What does the triangular structure of A say about the direction of information flow in a river?

B. Linearity and time invariance

4. Classifying systems. [Review Slides 7/51 and 14/51 in Module 4.pdf, uploaded to Brightspace]

A system is *linear* if its response to (x_0, u) satisfies superposition: the response to $(ax_0 + bz_0, au + bw)$ equals a times the response to (x_0, u) plus b times the response to (z_0, w) . It is *time invariant* if delaying the initial time and the input by τ simply delays the response by τ . Classify each system as linear or nonlinear, and as time invariant or time varying. Prove each property that holds, and give an explicit counterexample for each property that fails. Here u is the input, x the state, and y the output; when no output is given, take $y = x$.

- (a) $\dot{x} = -3x + 2u$, (b) $\dot{x} = -3x + 2u, \quad y = x + 5$,
 (c) $\dot{x} = -(1 + 0.5 \cos t)x + u$, (d) $x_{k+1} = 0.9x_k + u_k^2$,
 (e) $x_{k+1} = 0.9x_k + k u_k$, (f) $\dot{x} = -x + u, \quad y = \text{sat}(x) = \max\{-1, \min\{1, x\}\}$,
 (g) $\dot{x} = -x + xu$ (bilinear), (h) $y(t) = u(t - 2)$ (transport delay in a pipe).

For (c), solve the zero input response from $x(t_0) = 1$, and evaluate it one time unit later for $t_0 = 0$ and for $t_0 = \pi/2$. For (b), explain how a change of variables recovers a linear model; this is exactly what happens when we write models in deviation variables.

5. Superposition fails in an epidemic. [Review Slides 7/51 and 14/51 in Module 4.pdf, uploaded to Brightspace]

The SIR model with standard incidence is

$$\dot{S} = -\beta \frac{SI}{N}, \quad \dot{I} = \beta \frac{SI}{N} - \gamma I, \quad \dot{R} = \gamma I, \quad N = S + I + R,$$

with $\beta = 0.3 \text{ day}^{-1}$ and $\gamma = 0.1 \text{ day}^{-1}$. Let $x = (S, I, R)$ and write $\dot{x} = f(x)$.

- (a) Show that f is not linear by producing two specific states x, z with $f(x + z) \neq f(x) + f(z)$.
 (b) Show that $f(\alpha x) = \alpha f(x)$ for every $\alpha > 0$. Conclude that doubling every compartment doubles the whole trajectory. Why does this homogeneity still not make the model linear?
 (c) *Additivity experiment.* Town A starts at (9990, 10, 0) and town B at (10000, 0, 0). Simulate 200 days for each town separately and add the results. Then simulate one merged town, which corresponds to a new highway that mixes the two populations, starting from the sum of the initial states. Report the final number recovered and the peak number infected in both cases, and plot $I(t)$ for both. Interpret the result for transportation planning.
 (d) *Time shift experiment.* With constant β , the model is time invariant; explain why. Now suppose that a lockdown on calendar day 40 reduces β from 0.3 to 0.12 for the rest of the year. Seed 10 infections in a town of 10,000 on day 0, and separately on day 20. Compare the peak sizes and peak days, and plot both together with the day 0 curve shifted by 20 days. Is the model with the lockdown time invariant?
 (e) Linearize about the disease free state $(N, 0, 0)$ to show $\dot{I} \approx (\beta - \gamma)I$. Compute the early doubling time. Is this local model linear and time invariant, and for how long should it be trusted?

6. Linear or not, time varying or not: chemical reactors. [Review Slides 7/51, 14/51, and 15/51 in Module 4.pdf, uploaded to Brightspace]

- (a) *Fed batch reactor with a temperature ramp.* A first order reaction $A \rightarrow B$ runs while the jacket temperature is ramped, so the Arrhenius rate constant grows in time as $k(t) = k_0 e^{\theta t}$ with $k_0 = 0.05 \text{ min}^{-1}$ and $\theta = 0.02 \text{ min}^{-1}$. With feed rate $u(t)$ (mol/L/min),

$$\dot{C}_A = -k(t) C_A + u(t).$$

Using the solution

$$C_A(t) = \Phi(t, t_0) C_A(t_0) + \int_{t_0}^t \Phi(t, \tau) u(\tau) d\tau, \quad \Phi(t, \tau) = \exp\left(-\int_{\tau}^t k(s) ds\right),$$

prove that the system is linear. Then run the time shift test: with no feed, start at $C_A = 1 \text{ mol/L}$ at $t_0 = 0$ and, separately, at $t_0 = 30 \text{ min}$. Derive and evaluate $C_A(t_0 + 10)/C_A(t_0)$ in each case.

- (b) *Second order reaction.* An isothermal reaction $2A \rightarrow P$ obeys $\dot{C}_A = -kC_A^2$ with $k = 0.5$ L/(mol·min). Solve it analytically. Compare the responses from $C_0 = 1$ and $C_0 = 2$ mol/L, and show that the half life depends on C_0 . Is the system linear? Is it time invariant?
- (c) *CSTR with a conversion sensor.* An isothermal continuous stirred tank obeys $\dot{C} = (q/V)(C_{in} - C) - kC$, with inlet concentration C_{in} as the input. Is the state equation linear in (C, C_{in}) ? An online analyzer reports the conversion $y = 1 - C/C_{in}$. Is the output map linear? How would you obtain a linear model for control design?
- (d) Plot the two experiments of (a) and the two responses of (b), together with twice the $C_0 = 1$ response.

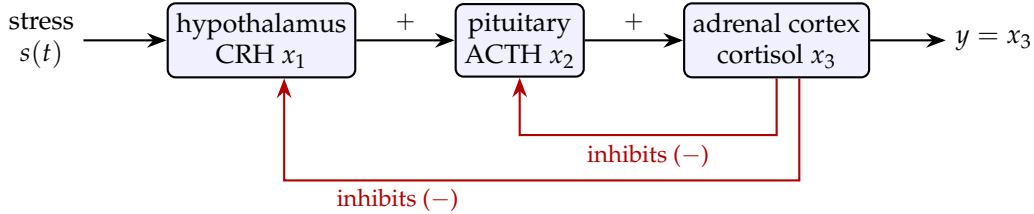
C. Linearization, discretization, validity, and local stability

7. The stress hormone axis (HPA). [Review Slides 19/51–26/51 in Module 4.pdf, uploaded to Brightspace]

The hypothalamic pituitary adrenal (HPA) axis regulates cortisol. Stress drives the hypothalamus to release CRH (x_1). CRH stimulates the pituitary to release ACTH (x_2), and ACTH stimulates the adrenal cortex to release cortisol (x_3). Cortisol inhibits both CRH and ACTH release (negative feedback). A normalized model, with time in hours, is

$$\dot{x}_1 = \frac{s}{1+x_3^2} - x_1, \quad \dot{x}_2 = \frac{x_1}{1+x_3^2} - x_2, \quad \dot{x}_3 = x_2 - x_3, \quad y = x_3,$$

where the stress drive $s(t) > 0$ is the input.

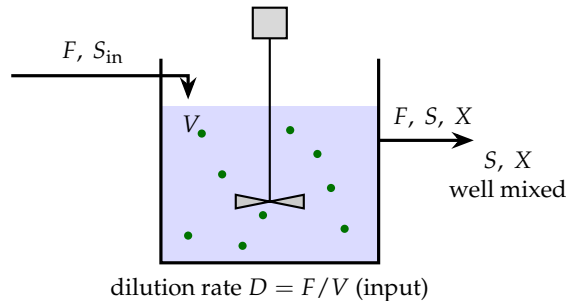


- Show that $\bar{x} = (2, 1, 1)$ is an equilibrium for $\bar{s} = 4$, and that it is the only equilibrium. *Hint:* show that every equilibrium satisfies $x_3(1+x_3^2)^2 = s$.
 - Linearize at $(\bar{x}, \bar{s})$ to obtain A , B , and C for the deviation variables.
 - Local stability.* Compute the eigenvalues of A . Is the local model stable? Report the period and the decay time constant of the oscillatory mode. Then compute the eigenvalues of the linearization for operating points $\bar{s} \in [0.25, 40]$. Does local stability depend on the operating point?
 - Discretization.* Compute the exact zero order hold (ZOH) model (A_d, B_d) using the block matrix exponential of Slide 20/51, and the forward Euler model, for $T_s = 0.25$ h and $T_s = 1$ h. Compare the eigenvalue magnitudes. Find the largest T_s for which the Euler model is stable, and explain why the ZOH model never loses stability.
 - Validity of the operating point.* Apply stress steps $s = 4(1 + \epsilon)$ with $\epsilon \in \{0.1, 0.5, 2\}$ from $\bar{x}$. Simulate the nonlinear and linear models for 20 h, and plot the cortisol responses. Compare the steady state cortisol from both models and report the error as a percentage of the true change. For what range of ϵ would you trust the local model?
8. **Activated sludge bioreactor (chemostat) and washout.** [Review Slides 19/51–26/51 in Module 4.pdf, uploaded to Brightspace]

A continuously fed bioreactor treats a wastewater stream. Biomass X (mg/L) grows on substrate S (mg/L) with Monod kinetics $\mu(S) = \mu_{\max} S / (K_s + S)$. The dilution rate $D = F/V$ (h^{-1}) is the control input, and the influent substrate S_{in} is a disturbance:

$$\dot{S} = D(S_{\text{in}} - S) - \frac{1}{Y}\mu(S)X, \quad \dot{X} = (\mu(S) - D)X.$$

Use $\mu_{\max} = 0.5 \text{ h}^{-1}$, $K_s = 50 \text{ mg/L}$, $Y = 0.5$, $\bar{S}_{\text{in}} = 500 \text{ mg/L}$, and the nominal dilution $\bar{D} = 0.25 \text{ h}^{-1}$.



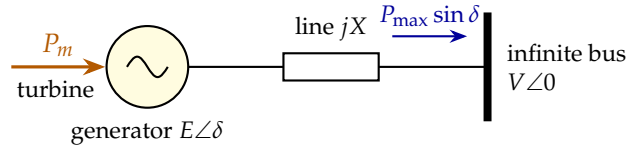
- Find both equilibria at $\bar{D}$: the operating point with $X > 0$, and the washout point with $X = 0$.
- Linearize at both equilibria to obtain A , $B = \partial f / \partial D$, and $E = \partial f / \partial S_{\text{in}}$. Note that D multiplies the states, so B depends on the operating point.
- Local stability.* Decide the stability of each local model from its eigenvalues. Explain in words why the washout point is unstable as long as $\mu(S_{\text{in}}) > D$.
- Discretization.* At the operating point, compute the ZOH and forward Euler models for $T_s = 0.5$ h and $T_s = 2$ h, and their eigenvalue magnitudes. What is the largest stable Euler step?
- Validity.* Starting at the operating point, step the dilution rate to $D \in \{0.30, 0.40, 0.47\} \text{ h}^{-1}$. Simulate both models for 300 h and plot X and S . Compare the steady states predicted by the linear model, $\delta x_{ss} = -A^{-1}B\delta D$, with the true nonlinear steady states. What does the local model completely miss at $D = 0.47 \text{ h}^{-1}$?

9. **Power grid swing equation.** [Review Slides 19/51–26/51 in *Module 4.pdf*, uploaded to Brightspace]

A synchronous generator connected to a large grid (an “infinite bus”) through a transmission line obeys the swing equation

$$M\ddot{\delta} + D_d\dot{\delta} = P_m - P_{\max} \sin \delta,$$

where δ is the rotor angle relative to the grid (rad), $\omega = \dot{\delta}$ is the speed deviation (rad/s), P_m is the mechanical power (the input), and $P_{\max} \sin \delta$ is the electrical power sent to the grid. Use $M = 0.02 \text{ pu}\cdot\text{s}^2/\text{rad}$, $D_d = 0.02 \text{ pu}\cdot\text{s}/\text{rad}$, $P_{\max} = 2 \text{ pu}$, and $\bar{P}_m = 1 \text{ pu}$.



- Write the model $\dot{x} = f(x, u)$ with $x = (\delta, \omega)$ and $u = P_m$. Find both equilibria in $[0, \pi]$ for $\bar{P}_m = 1$.
- Linearize at both equilibria. Explain why $P_{\max} \cos \bar{\delta}$ is called the synchronizing coefficient.
- Local stability.* Classify each equilibrium from its eigenvalues. For the stable one, report the natural frequency (Hz) and the damping ratio of the electromechanical oscillation.
- Discretization.* At the stable equilibrium, compute the ZOH and forward Euler models for $T_s = 0.01$ s and $T_s = 0.02$ s. Show that Euler is stable only for $T_s < 2|\text{Re } \lambda|/|\lambda|^2$, and evaluate this bound. What does this tell you about using forward Euler on lightly damped oscillatory systems?
- Validity.* A fault is cleared at $t = 0$, leaving $\delta(0) = \bar{\delta}$ but a speed deviation $\omega(0) \in \{2, 8, 14\} \text{ rad/s}$. Simulate the nonlinear and linear models for 3 s and plot $\delta(t)$ in degrees. For which disturbance does the linear model give a qualitatively wrong answer, and what happens physically to the generator?

D. Solving the state evolution

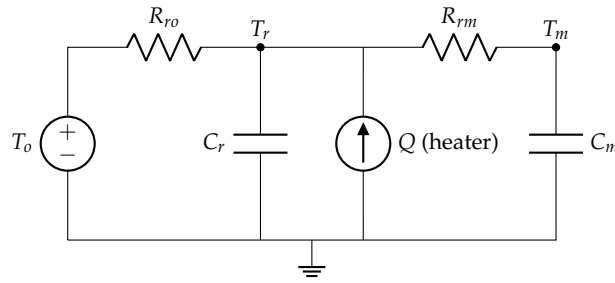
10. **Age structured fish population (discrete time).** [Review Slides 16/51, 18/51, and 25/51–26/51 in Module 4.pdf, uploaded to Brightspace]

A managed fishery tracks juveniles J_k and adults A_k (thousands of fish) at the start of year k . Each adult produces 0.7 surviving juveniles per year, 20% of juveniles survive to become adults, and 50% of adults survive to the next year. A hatchery stocks u_k (thousands) juveniles per year:

$$J_{k+1} = 0.7A_k + u_k, \quad A_{k+1} = 0.2J_k + 0.5A_k.$$

- Write $x_{k+1} = Ax_k + Bu_k$ with $x_k = (J_k, A_k)$. Find the eigenvalues and eigenvectors of A .
 - Diagonalize A to derive a closed form expression for A^k . Verify that it reproduces A for $k = 1$ and A^5 computed numerically.
 - No stocking.* With $u_k = 0$ and $x_0 = (400, 300)$, write x_k in modal form and compute x_1, x_2, x_3 both from the formula and by recursion. What fraction of the population remains after 10 years?
 - Constant stocking.* With $u_k = 100$ and a depleted initial stock $x_0 = (50, 20)$, find the equilibrium $\bar{x}$ and write $x_k = \bar{x} + A^k(x_0 - \bar{x})$. Compute x_3 both by recursion and with the convolution sum $x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} B u_i$. After which year do both classes stay within 5% of $\bar{x}$?
 - Plot both scenarios for 20 years. Explain the role of the negative eigenvalue. Find the fecundity (currently 0.7) at which the population would be exactly self sustaining, meaning that the dominant eigenvalue equals 1.
11. **Warming up a room with thermal mass (continuous time, real modes).** [Review Slides 9/51, 15/51, and 16/51 in Module 4.pdf, uploaded to Brightspace]

A room is modeled by the thermal circuit below. The room air (capacity C_r) exchanges heat with the building's thermal mass (capacity C_m) through R_{rm} , and with the outdoors through R_{ro} . A heater injects Q kW into the air. Outdoor temperature is constant at $T_o = 5^\circ\text{C}$. Use $C_r = 2 \text{ kWh}/^\circ\text{C}$, $C_m = 4 \text{ kWh}/^\circ\text{C}$, $R_{rm} = 1.25^\circ\text{C}/\text{kW}$, and $R_{ro} = 1^\circ\text{C}/\text{kW}$.



- Write an energy balance for each capacitance. With $x = (T_r - T_o, T_m - T_o)$ and $u = Q$, derive $\dot{x} = Ax + Bu$ and show that T_o drops out.
 - Compute the eigenvalues and eigenvectors of A . Use diagonalization to derive e^{At} in closed form, and check it against expm at $t = 2$ h.
 - The heater is switched on with $Q = 15 \text{ kW}$ at $t = 0$ from a cold start ($x(0) = 0$). Use the state transition formula to derive $T_r(t)$ and $T_m(t)$ in closed form.
 - Report the two time constants. When does the room air first reach 18°C ? Plot both temperatures for 40 h. Explain physically why the air rises quickly at first and then slowly, using the two mode shapes.
12. **A lightly damped mode (continuous time, complex modes).** [Review Slides 15/51, 16/51, and 17/51 in Module 4.pdf, uploaded to Brightspace]

Consider the generic system

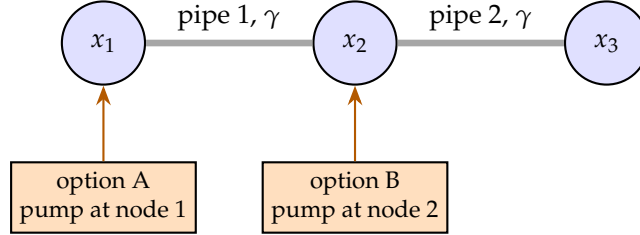
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u(t) = 10 \text{ for } t \geq 0.$$

- (a) Find the eigenvalues, the natural frequency ω_n , and the damping ratio ζ .
- (b) Let $N = A + I$. Show that $N^2 = -4I$, and use this fact as on Slide 17/51 to derive $e^{At} = e^{-t}(I \cos 2t + \frac{1}{2}N \sin 2t)$. Check that $e^{A \cdot 0} = I$ and that $\frac{d}{dt}e^{At}|_{t=0} = A$.
- (c) Compute the natural response $e^{At}x(0)$ and the forced response separately. *Hint:* for invertible A and constant u , $\int_0^t e^{A(t-\tau)} B d\tau u = A^{-1}(e^{At} - I)Bu$.
- (d) Give the total $x(t)$. Find the peak time and peak value of $y(t)$ and the time to settle, using $4/|\operatorname{Re} \lambda|$. Plot the natural part, the forced part, and the total response.

E. Controllability

13. **Where to put the pump in a three tank network.** [Review Slides 13/51, 27/51–30/51, 33/51, and 48/51 in Module 4.pdf, uploaded to Brightspace]

Three storage tanks are connected in a line by two pipes of conductance $\gamma = 0.2$. With incidence matrix M (one column per pipe), the sampled network obeys $x_{k+1} = (I - \gamma MM^T)x_k + Bu_k$, where x is the vector of tank levels in deviation from nominal. The utility can afford one booster pump.



- Write M , compute $A = I - \gamma MM^T$, and find its eigenvalues and eigenvectors. Interpret the eigenvalue 1.
 - Option A.* With $B = e_1$, form the controllability matrix. Is (A, B) controllable?
 - Option B.* With $B = e_2$, compute the rank of the controllability matrix. Use the PBH test to identify the uncontrollable mode, and explain the result physically using the symmetry of the network. Is the pair stabilizable?
 - With option A, find the three step pumping sequence u_0, u_1, u_2 that moves the levels from $x_0 = 0$ to $x_3 = (1, 0, -1)$. Why is this target impossible with option B?
 - Compute the 10-step Gramian $W_c(10) = \sum_{i=0}^9 A^i B B^T (A^T)^i$ for option A and for two pumps at nodes 1 and 3. Report $\lambda_{\min}(W_c)$ and the minimum energy $x_f^T W_c^{-1} x_f$ needed to reach $x_f = (1, 0, -1)$. Which design would you recommend, and why?
 - If pipe 2 is closed for repairs, is the network still controllable from node 1?
14. **Controllability, stabilizability, and the cost of weak coupling.** [Review Slides 28/51, 29/51, 31/51, 33/51, and 34/51 in Module 4.pdf, uploaded to Brightspace]

Consider the generic system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $\alpha \in \mathbb{R}$ is a coupling parameter.

- Compute the controllability matrix as a function of α , and find all α for which (A, B) is controllable.
- For $\alpha = 0$, apply the PBH test at every eigenvalue. Which mode is uncontrollable? Is the system stabilizable?
- For $\alpha = 1$, move the actuator to the first state, $B' = e_1$. Is (A, B') controllable? Is it stabilizable? If so, find a gain K that makes $A - B'K$ stable.
- For $\alpha = 1$ and the original B , use coefficient matching on $\det(\lambda I - A + BK)$ to find $K = [k_1 \ k_2 \ k_3]$ that places the closed loop poles at $-2, -3, -4$. Verify your answer with pplace.
- Repeat (d) for a general $\alpha \neq 0$, and show that $k_1 = 20/\alpha$. What happens to the initial control effort from $x_0 = e_1$ as $\alpha \rightarrow 0$? Relate this to the statement “rank answers yes or no; how hard is a separate question.”

F. Observability

15. **Where to put the dissolved oxygen sensor.** [Review Slides 35/51, 36/51, 37/51, and 48/51 in Module 4.pdf, uploaded to Brightspace]

Return to the two reach river of Problem 3 (see its figure). With $x = (L_1, D_1, L_2, D_2)$, the model is $\dot{x} = Ax$ (inputs are known and can be removed) with

$$A = \begin{bmatrix} -0.8 & 0 & 0 & 0 \\ 0.3 & -1.1 & 0 & 0 \\ 0.4 & 0 & -0.7 & 0 \\ 0 & 0.4 & 0.3 & -1.0 \end{bmatrix} \text{ day}^{-1}.$$

- A DO (deficit) sensor at the end of reach 2 gives $C = [0 \ 0 \ 0 \ 1]$. Form the observability matrix, find its rank, and confirm your answer with the PBH test at every eigenvalue.
- The sensor is moved upstream to reach 1, $C = [0 \ 1 \ 0 \ 0]$. Find the rank and the unobservable subspace, and explain physically why this subspace cannot be seen. Is the pair detectable?
- Instead, a BOD analyzer is installed in reach 2, $C = [0 \ 0 \ 1 \ 0]$. Find the rank and the unobservable subspace, and explain.
- For the reach 2 deficit sensor, compute the observability Gramian

$$W_o(T) = \int_0^T e^{A^T \tau} C^T C e^{A \tau} d\tau$$

for $T = 10$ days numerically. Report its eigenvalues and the least observable direction, and compare with a design that measures both D_1 and D_2 . Explain the difference between “observable” and “well observable.”

16. **Epidemic surveillance: what can case counts tell us?** [Review Slides 36/51 and 37/51 in Module 4.pdf, uploaded to Brightspace]

The SEIR model tracks susceptible, exposed (infected but not yet infectious), infectious, and recovered people. Near the disease free state $S \approx N$, the infection dynamics are approximately

$$\dot{E} = \beta I - \sigma E, \quad \dot{I} = \sigma E - \gamma I, \quad \dot{R} = \gamma I,$$

with $\sigma = 0.2 \text{ day}^{-1}$ (5-day incubation), $\gamma = 1/7 \text{ day}^{-1}$ (7-day infectious period), and $\beta = 0.3 \text{ day}^{-1}$.

- Write A for $x = (E, I, R)$, compute its eigenvalues and the basic reproduction number $R_0 = \beta/\gamma$, and decide the stability of the disease free state. Compute the doubling time of the growing mode.
- Clinical case counts*, $y = I$. Form the observability matrix and find its rank and the unobservable subspace. Apply the PBH test at each eigenvalue. Is the pair detectable?
- Serology survey*, $y = R$. Is the system observable?
- Wastewater surveillance*, $y = 0.5E + I$, since exposed people also shed virus. Is the system observable?
- Explain why the recovered compartment is invisible to sensors that only see E and I . Does this matter for predicting $I(t)$ with the linear model? Why does it matter in the full nonlinear model, where $S = N - E - I - R$?

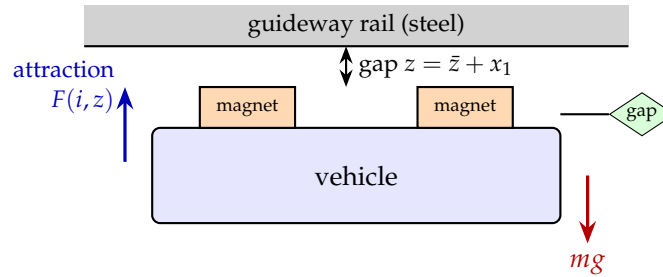
G. Observer based control

17. **Levitating a maglev vehicle with only a gap sensor.** [Review Slides 31/51, and 38/51–41/51 in Module 4.pdf, uploaded to Brightspace]

An electromagnet lifts a maglev vehicle toward a guideway rail. The magnetic attraction grows as the gap shrinks, so the equilibrium gap is unstable. In normalized units, the linearized gap dynamics are

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

where x_1 is the gap deviation, x_2 its rate, and u the change in magnet current. Only the gap is measured.



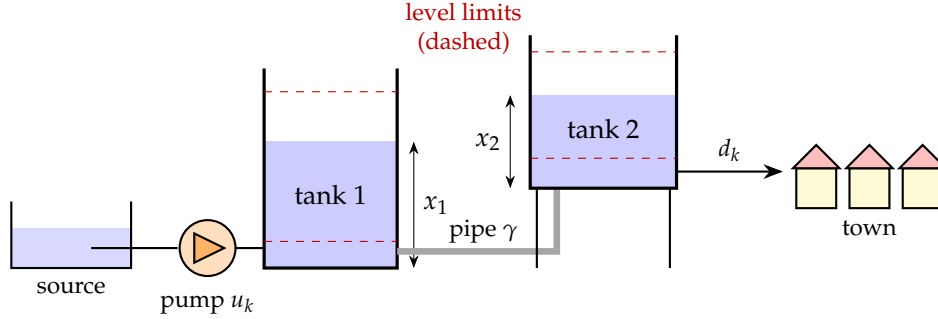
- Find the open loop eigenvalues. Check controllability and observability.
- Use coefficient matching to find the state feedback gain K that places the controller poles at $-4, -5$.
- Use coefficient matching to find the observer gain L that places the observer poles at $-12, -15$.
- Write the closed loop of $u = -K\hat{x}$ and $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ in the coordinates (x, e) . Show that it is block triangular, and list its eigenvalues.
- Simulate 4 s from $x(0) = (1, 0)$ with the observer started at $\hat{x}(0) = 0$. Plot the gap, the true and estimated states, and the input, and compare with full state feedback $u = -Kx$. What is the peak $|u|$ in each case, and why is it larger with the observer?
- Redesign the observer with slow poles $-1, -1.5$ and repeat the simulation. Is the closed loop still stable? Discuss the tradeoff in choosing observer speed when the gap sensor is noisy.

H. Model predictive control (mandatory)

18. **MPC for a water network with time of use electricity prices.** [Review Slides 13/51, 18/51, and 42/51–46/51 in Module 4.pdf, uploaded to Brightspace]

A pump station lifts water into tank 1, which feeds an elevated tank 2 through a pipe; the town draws its demand from tank 2. With hourly samples, levels $x_k = (x_{1,k}, x_{2,k})$ in meters, pumping u_k (m/h of level in tank 1), and demand d_k (m/h of level in tank 2),

$$x_{k+1} = \underbrace{\begin{bmatrix} 1-\gamma & \gamma \\ \gamma & 1-\gamma \end{bmatrix}}_A x_k + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_B u_k + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_E d_k, \quad \gamma = 0.4.$$



The forecast demand is $\hat{d}_k = 0.5 + 0.25 \sin(2\pi(k-6)/24)$. The actual demand equals the forecast except during an unforecast heat wave, when it is 15% higher for $30 \leq k < 42$. With the hour of day $h = k \bmod 24$, the electricity price is $c_k = 0.5$ for $h < 7$ or $h \geq 22$, $c_k = 3$ for $14 \leq h < 20$, and $c_k = 1$ otherwise. The constraints are

$$1 \leq x_1 \leq 5.5, \quad 1.5 \leq x_2 \leq 4.5, \quad 0 \leq u_k \leq 1.2.$$

At time k , MPC minimizes over the horizon N

$$J_k = \sum_{i=0}^{N-1} \left[(x_{i+1} - r)^T Q (x_{i+1} - r) + R u_i^2 + c_{k+i} u_i \right], \quad Q = \text{diag}(0.1, 1), \quad R = 0.2, \quad r = (4.25, 3),$$

using the forecast $\hat{d}$. Start from $x_0 = (2, 2)$ and simulate 72 hours.

- Verify that each row of the model is a node balance. Find the eigenvalues of A and interpret the eigenvalue 1: show that total storage obeys $s_{k+1} = s_k + u_k - d_k$. Show that $r = (4.25, 3)$ is consistent with steady pumping at the mean demand 0.5.
- Following Slide 43/51, write the stacked prediction $X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{E}\hat{D}$ for $N = 3$ with numerical matrices.
- Following Slide 44/51, derive the condensed QP $\min_U \frac{1}{2}U^T H U + f^T U$ subject to $GU \leq g + Sx_0$. Give H , f , G , g , and S in terms of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, $\bar{Q} = I_N \otimes Q$, $\bar{r}$, $\hat{D}$, and the price vector c . Is the QP convex?

Kronecker product. For $P \in \mathbb{R}^{m \times n}$ with entries p_{ij} and any matrix $Q \in \mathbb{R}^{p \times q}$, the Kronecker product $P \otimes Q \in \mathbb{R}^{mp \times nq}$ is the block matrix obtained by replacing each entry p_{ij} with the block $p_{ij}Q$:

$$P \otimes Q = \begin{bmatrix} p_{11}Q & \cdots & p_{1n}Q \\ \vdots & \ddots & \vdots \\ p_{m1}Q & \cdots & p_{mn}Q \end{bmatrix}.$$

In particular, $I_N \otimes Q = \text{diag}(Q, \dots, Q)$ repeats Q N times along the block diagonal, so $\bar{Q}$ weights every predicted state $x_1, \dots, x_N$ in X by the same Q . For example,

$$I_2 \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

In MATLAB this is `kron(eye(N),Q)`; in Python, `numpy.kron(np.eye(N), Q)`. Similarly, $\bar{r} = \mathbf{1}_N \otimes r$ stacks r N times.

- (d) Implement receding horizon MPC with $N = 12$ in CVX/CVXPY (or quadprog). At each hour, solve the QP from the measured x_k , apply only u_0^* , and advance the plant with the *actual* demand. If a QP is infeasible, apply the fallback $u_k = u_{\max}$ if $x_{2,k} < 3$ and $u_k = 0$ otherwise, and count the event. Report the first optimal plan and u_0^* , and make a three panel plot of the levels with their bounds, pumping and demand, and price.
- (e) *Rule based control.* Most utilities still run pumps with simple level rules in their SCADA system. Implement the following on/off rule with a hysteresis band on the tank 2 level and overflow protection on tank 1. Start with the pump ON, and at every hour k :
- if $x_{2,k} \leq 2.5$, switch the pump ON; if $x_{2,k} \geq 3.5$, switch it OFF; otherwise keep the previous pump status;
 - apply $u_k = u_{\max} = 1.2$ when the pump is ON and $x_{1,k} < 5.2$, and $u_k = 0$ otherwise. The overflow check sets $u_k = 0$ but does not change the pump status.

Simulate the rule with the same actual demand and initial state. For both MPC and the rule, tabulate the energy cost $\sum_k c_k u_k$; the tracking cost $\sum_k (x_{k+1} - r)^T Q (x_{k+1} - r)$; the total closed loop cost (tracking + $R \sum_k u_k^2$ + energy); the total pumped volume; the regulation metrics $\text{RMS}_{k \geq 6}(x_{2,k} - 3)$ and $\max_{k \geq 6} |x_{2,k} - 3|$, which exclude the common startup transient; the minimum levels; and any constraint violations. Confirm, or refute, that the rule based controller is worse than MPC both in cost and in regulation. Add the rule based trajectories to your plot.

- (f) Repeat the MPC for $N \in \{2, 4, 8, 12, 24\}$ and tabulate the same metrics and the number of infeasible solves. Explain what goes wrong with a very short horizon. *Hint*: how many steps does it take before u can affect x_2 ?
- (g) Discuss: when does MPC choose to pump, and why? Why does the rule based controller pay more even though it pumps about the same volume? How does feedback handle the heat wave that the forecast missed? How would soft constraints (Slide 46/51) change the formulation?