

Module 05

Drinking Water Distribution Networks

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**CE 4240/5240 — Introduction to
Infrastructure Systems Engineering**

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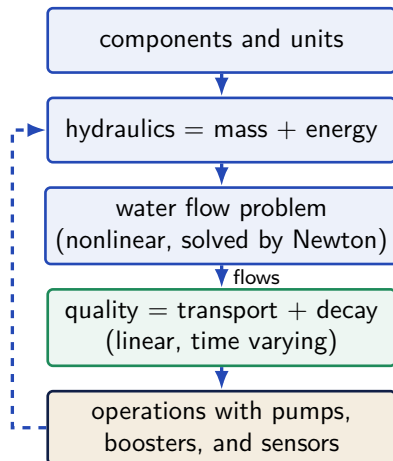
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Outline and objectives

- ➊ Explain what a water distribution network (WDN) must deliver and why operating one is hard
- ➋ Name the network components and convert between head, pressure, and flow units
- ➌ Write the mass and energy balance of every component
- ➍ Solve the water flow problem for a looped network and simulate a day of operation
- ➎ Model chlorine transport, mixing, and decay as a linear state space model
- ➏ Pose pump scheduling, chlorine dosing, and booster and sensor placement as optimization and control problems



Notation and nomenclature...

for the module. Go back go it when in doubt or need...

Hydraulics

Symbol	Meaning (US units)
h_i, z_i	head, elevation at node i (ft)
p_i	pressure at node i (psi)
q_{ij}	flow on link $i \rightarrow j$ (GPM)
d_i	demand at junction i (GPM)
$\Delta h_{ij}^P, \Delta h_{ij}^M$	pipe head loss, pump head gain
R, μ	pipe resistance, flow exponent
h_0, r, ν	pump shutoff head, curve coefficients
s	relative pump speed, $0 \leq s \leq 1$
β	tank head change per gallon (ft/gal)
M	node by link incidence matrix
$k, \Delta t$	time step index, step length

Water quality

Symbol	Meaning (mg/L, 1/day)
c_i^J, c_i^{TK}, c_i^R	junction, tank, reservoir chlorine
$c_{ij}^P(s, k)$	chlorine, segment s of pipe ij
c^B, q^B	booster concentration and flow
V^{TK}	tank volume (gal)
v_{ij}	velocity in pipe ij (ft/s)
Δx	pipe segment length (ft)
$\alpha = v\Delta t/\Delta x$	Courant number (upwind weight)
$K_b, \kappa = K_b\Delta t$	decay rate, decay per step
χ, u	quality state, booster input

Superscripts J, TK, R, P, M, V, B mark junction, tank, reservoir, pipe, pump, valve, booster.

PART 1

Why water networks

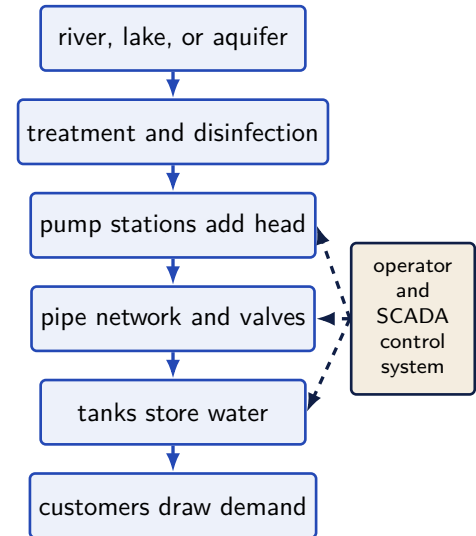
What a water network must deliver, the problems we pose on it, and why they are hard

What a water network must deliver

- The **purpose** is enough water, at enough pressure, safe to drink, for every customer, every hour
- The **scale** is about 2.2 million miles of buried US pipe system (bad news, a water main breaks every few minutes!!!)
- **Energy** for pumping is a utility's largest energy bill, and water and wastewater systems use about 2% of US energy (already surpassed by greedy data centers sadly)
- **Pain points** include aging pipes, shifting demand, drought, extreme weather, climate change, stricter regulations, rules, etc...

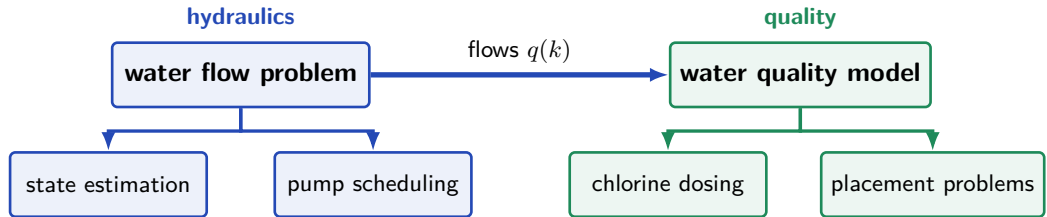
Water utility's job

Choose pump, valve, and chlorination actions so that every node gets enough pressure and safe water, at least cost, despite uncertain demand.



SCADA (supervisory control and data acquisition) is the utility's sensor and command network.

Problems we pose on a water network



Problem	Question	Mathematical tool
Water flow problem	heads and flows, given demands and pump settings	nonlinear equations, Newton (Module 3)
State estimation	heads and flows from a few pressure sensors	least squares, a QP (Module 3)
Pump scheduling	cheapest pump plan that keeps tanks and pressures in range	MINLP, or LP and QP after approximation
Quality model	chlorine everywhere, given flows and doses	linear time varying state space (Module 4)
Chlorine dosing	booster injections that keep the residual in range	LP or QP, MPC (Module 4)
Placement	where to inject and where to measure	mixed integer, controllability and observability

Hydraulics feed quality so every quality problem needs the flows first.

Six technical challenges

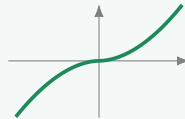
Unknown flow direction

Flows reverse during the day, so signs are unknown before solving



Nonlinear, nonconvex

Head loss $\propto q|q|^{0.852}$ and pump curves give nonconvex equality constraints



Topology

Loops, rings, and trees with thousands of nodes and links



Uncertainty

Demand, pipe roughness, and sensor noise follow unknown distributions



Transport PDEs

Chlorine obeys an advection reaction PDE in every pipe, so it must be discretized

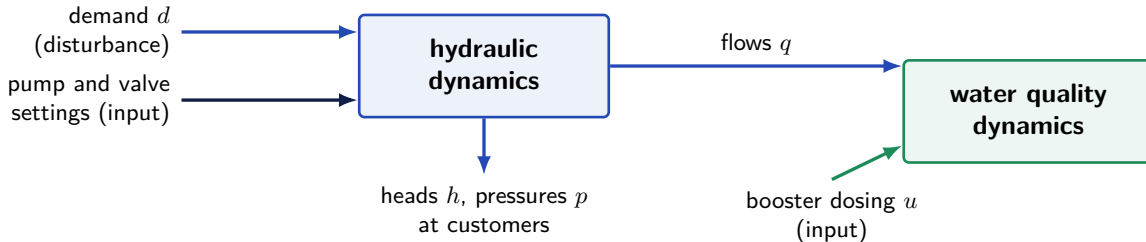


Coupled problems

Quality depends on hydraulics, so joint pump and booster control is large and nonconvex



Two coupled dynamic models



	Hydraulics	Water quality
States	tank heads (memory)	chlorine in junctions, tanks, pipe segments
Algebraic variables	junction heads, link flows	junction concentrations
Inputs	pump speed or status, valve settings	booster injections
Model type	nonlinear difference algebraic equations	linear, time varying difference equations
Time step	15 min to 1 h	seconds to minutes

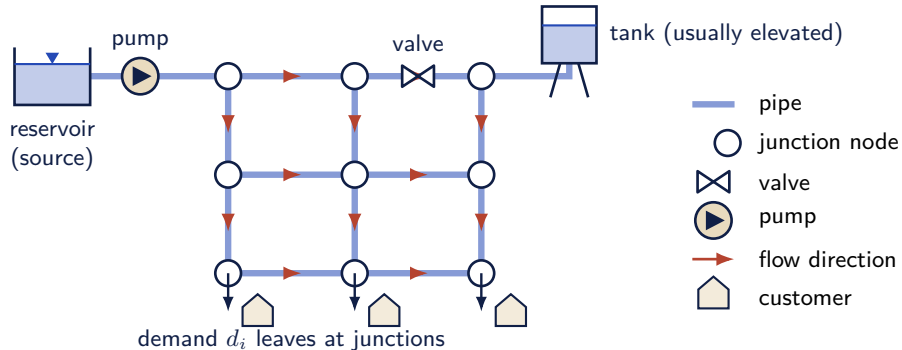
The goal is to simulate both models from topology, parameters, demands, and controls.

PART 2

Network components

Nodes and links, head and pressure, and the network as a graph

Network components



Nodes (carry a head h_i)

- **Junctions** join pipes, release demand, and store nothing
- **Tanks** store water, so their level changes over time
- **Reservoirs** have a fixed head and unlimited supply (lake, aquifer, treatment plant clearwell)

Links (carry a flow q_{ij})

- **Pipes** lose head to friction
- **Pumps** add head at fixed or variable speed
- **Valves** include pressure reducing (PRV), flow control (FCV), throttle (TCV), and one way check valves

A small town has hundreds of junctions and a large city has 10^5 to 10^6 .

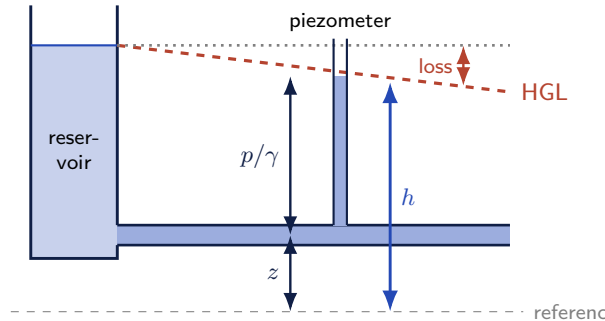
Head and pressure

Hydraulic head is energy per unit weight of water, measured in feet

$$h = z + \frac{p}{\gamma}$$

$$\gamma = \rho g = 62.4 \text{ lb/ft}^3$$

- z is the elevation above a reference
- p/γ is the pressure head
- $1 \text{ psi} = 144/62.4 = 2.31 \text{ ft of water}$
- Velocity head $v^2/2g$ is dropped, since at 1.25 ft/s it is only 0.02 ft



The HGL (hydraulic grade line) is the head along the pipe, and it falls in the flow direction by the friction loss.

Water flows from high head to low head, not from high pressure to low pressure.

Pressure from head, and US units

Example. Junction J4 of the coming example has head 909.03 ft and sits at $z = 800$ ft, so

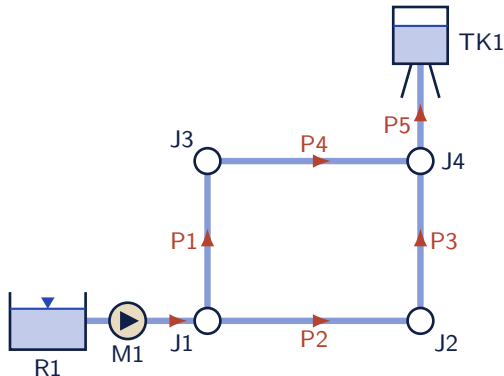
$$p = \frac{h - z}{2.31} = \frac{909.03 - 800}{2.31} = \boxed{47.2 \text{ psi}}$$

- The usual service range is 40 to 80 psi
- Never below 20 psi
- As a rule of thumb, every 2.31 ft of height is 1 psi, so a customer 50 ft uphill loses about 22 psi

US unit	Equivalent
1 ft of head	0.433 psi = 0.3048 m
1 psi	2.31 ft = 6.895 kPa
1 GPM	0.0631 L/s
1 cfs (ft ³ /s)	448.8 GPM
1 ft ³	7.48 gal
1 gal	3.785 L

I used SI units in Module 3, but US utilities and EPANET's US mode use ft, psi, and GPM, and this module follows them (sorry!!).

The network as a graph, building the adjacency/incidence matrix!



2 fixed head nodes (R1, TK1), 4 junctions, 6 links. Arrows are *assumed* directions.

The incidence matrix M has +1 at the start node of a link and -1 at its end node (the convention of Module 4)

$$\begin{array}{c}
 \begin{array}{c} \text{R1} \\ \text{J1} \\ \text{J2} \\ \text{J3} \\ \text{J4} \\ \text{TK1} \end{array}
 \begin{bmatrix}
 \text{M1} & \text{P1} & \text{P2} & \text{P3} & \text{P4} & \text{P5} \\
 \begin{array}{cccccc}
 1 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 1 & 1 & 0 & 0 & 0 \\
 0 & 0 & -1 & 1 & 0 & 0 \\
 0 & -1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & -1 & -1 & 1 \\
 0 & 0 & 0 & 0 & 0 & -1
 \end{array}
 \end{bmatrix}
 \end{array}
 \quad M_J$$

- **Mass** at junctions (dashed block M_J) gives $M_J q + d = 0$
Row J2 reads $-q_{P2} + q_{P3} + d_2 = 0$
- **Energy** on every link uses $(M^T h)_\ell = h_{\text{start}} - h_{\text{end}}$, the head drop along link ℓ
- There are 6 flows and 4 junction heads, and 4 mass and 6 energy equations

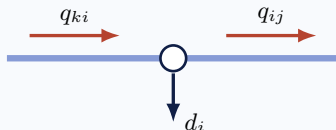
PART 3

Hydraulic models

Mass and energy balance for pipes, pumps, tanks, and valves, then a first network solved by hand

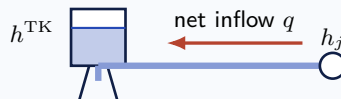
Mass and energy balance in each component

Junction mass balance (linear)



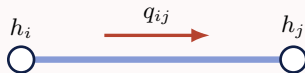
$$\sum_{\text{in}} q_{ki} - \sum_{\text{out}} q_{ij} = d_i$$

Tank mass balance (linear)



$$h^{\text{TK}}(k+1) = h^{\text{TK}}(k) + \beta \Delta t q(k)$$

Pipe energy balance (nonlinear)



$$h_i - h_j = \Delta h_{ij}^P = R q_{ij} |q_{ij}|^{\mu-1}$$

Reservoir and pump energy (nonlinear)



$$h_j - h_i^R = \Delta h_{ij}^M = s^2 (h_0 - r(q_{ij}/s)^\nu)$$

$\beta = 1/(7.48 S)$ converts gallons to feet of level for a tank of cross section S (ft²). A reservoir has a fixed head, so it adds no equation and no unknown, i.e., its variable $h^R = \text{constant}$ all the time.

Mass balance is linear in the flows but sadly the energy balance is nonlinear and nonconvex in the flows.

Pipe head loss and the Hazen-Williams model

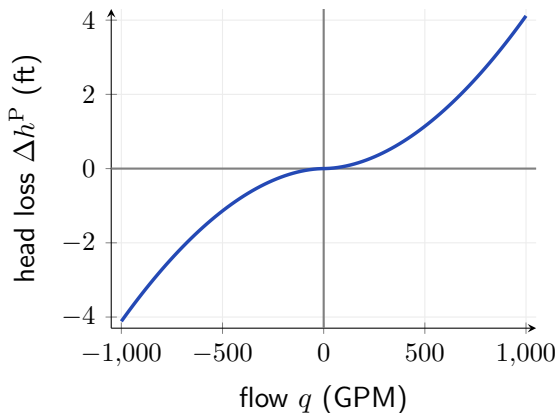
For a pipe of length L and diameter D (ft), with flow in cfs,

$$\Delta h^P = \frac{4.727 L}{C^{1.852} D^{4.871}} q^{1.852}$$

C is the roughness coefficient, about 140 for new plastic pipe and about 100 for aged cast iron (why your omelette sticks on cast iron pan, hehe).

Allowing either flow sign gives

$$\Delta h_{ij}^P = R_{ij} q_{ij} |q_{ij}|^{\mu-1}, \quad \mu = 1.852$$



The 1000 ft, 12 in, $C = 100$ pipe of the worked example.

- Odd and increasing, so reversing the flow reverses the loss
- Slope $\mu R |q|^{\mu-1}$ is zero at $q = 0$, which troubles Newton's method later

EPANET also offers Darcy-Weisbach ($\mu = 2$ with a flow dependent friction factor) and Chezy-Manning.

Example on pipe resistance and pipe size

Compute R for $L = 1000$ ft, $D = 12$ in = 1 ft, $C = 100$, with flow in GPM (1 cfs = 448.8 GPM)

$$R = \frac{4.727 \cdot 1000}{100^{1.852} \cdot 1^{4.871} \cdot 448.8^{1.852}}$$

$$= \boxed{1.144 \times 10^{-5}}$$

At $q = 440.6$ GPM the loss is

$$\Delta h^P = 1.144 \times 10^{-5} \times 440.6^{1.852} = 0.90 \text{ ft}$$

Same flow, other diameters (1000 ft, $C = 100$, $q = 440.6$ GPM)

D (in)	R	Δh^P (ft)
6	3.35×10^{-4}	26.4
8	8.25×10^{-5}	6.50
12	1.14×10^{-5}	0.90
16	2.82×10^{-6}	0.22

- Doubling the flow multiplies the loss by $2^{1.852} = 3.6$
- Doubling the diameter divides it by $2^{4.871} = 29$ so the pipe **diameter is the strongest design lever**

Pump head gain and characteristic curves

A pump adds head that falls as flow rises

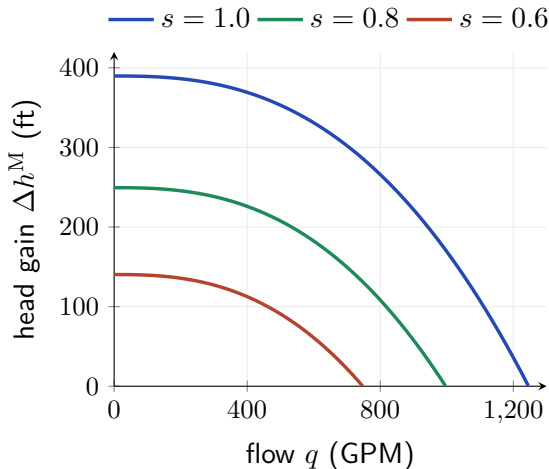
$$\Delta h^M(q) = s^2 \left(h_0 - r \left(\frac{q}{s} \right)^\nu \right)$$

- h_0 is the shutoff head at zero flow, and r, ν are fitted to the manufacturer's curve
- A fixed speed pump has $s = 1$ and is either on or off
- A variable speed pump uses $s \in [s_{\min}, 1]$ as a continuous control input

Power drawn with efficiency η is

$$P [\text{kW}] = \frac{0.746 q [\text{GPM}] \Delta h^M [\text{ft}]}{3960 \eta}$$

For example, $q = 921$ GPM, $\Delta h^M = 210$ ft, and $\eta = 0.75$ give $P = 48.6$ kW, about 1170 kWh per day.



$h_0 = 389.72$ ft, $r = 3.746 \times 10^{-6}$, $\nu = 2.59$, the pump of the worked example.

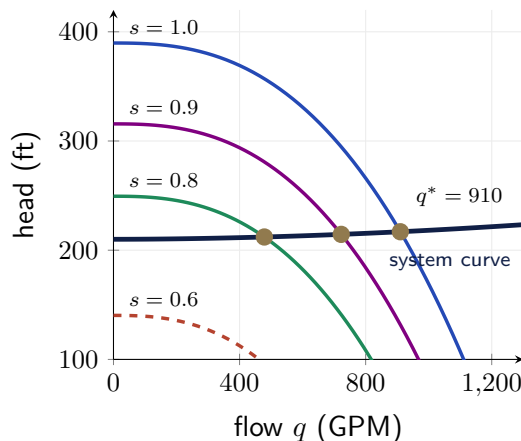
Example on where the pump operates

Reservoir at 700 ft → pump → 2000 ft of 12 in pipe ($R = 2.288 \times 10^{-5}$) → tank at head 910 ft. Energy along the path gives

$$\underbrace{s^2(h_0 - r(q/s)^\nu)}_{\text{pump curve}} = \underbrace{210 + Rq^{1.852}}_{\text{system curve}}$$

speed s	flow q^* (GPM)	head Δh^M (ft)
1.0	910	216.9
0.9	723	214.5
0.8	479	212.1
0.6	0	no flow

- 20% slower gives 47% less flow
- At $s = 0.6$ the pump cannot beat the 210 ft lift



The operating point is where the head the pump supplies equals the head the system demands.

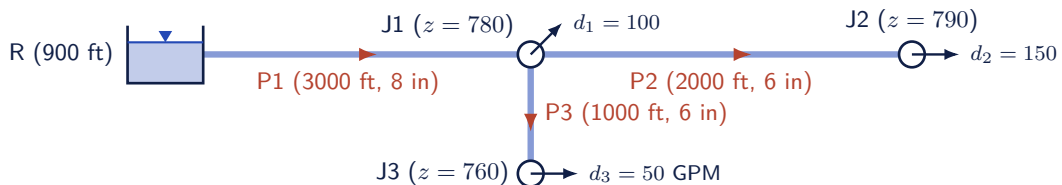
Valves and the component summary

Component	Kind	Hydraulic model	Character
Junction	node	$\sum q_{\text{in}} - \sum q_{\text{out}} = d_i$	linear, algebraic
Tank	node	$h^{\text{TK}}(k+1) = h^{\text{TK}}(k) + \beta \Delta t (q_{\text{in}} - q_{\text{out}})$	linear, dynamic
Reservoir	node	$h = h^{\text{R}}$ (fixed)	boundary condition
Pipe	link	$h_i - h_j = R q q ^{0.852}$	nonlinear
Pump	link	$h_j - h_i = s^2 (h_0 - r(q/s)^\nu)$, or $q = 0$ if off	nonlinear, switching
TCV (open)	link	$h_i - h_j = K_v q q $ (minor loss)	nonlinear
Check valve	link	$q \geq 0$, closes on reverse flow	switching
PRV	link	$h_j = h_{\text{set}}$ when active, else open or closed	switching
FCV	link	$q = q_{\text{set}}$ when active	switching

- A PRV protects a low lying zone by throttling until the downstream head equals its setting. A check valve stops a tank from draining backward through an idle pump
- Switching elements turn one model into several, and on/off choices make optimization mixed integer (Module 3)

Pipes and pumps make the equations nonlinear, and valves and pumps make them switch.

Example of a branched network that needs no solver



All pipes have $C = 100$. The task is to find every head and pressure.

Step 1. Flows from mass alone. Each pipe of a tree carries all the demand downstream of it

$$q_{P2} = 150, \quad q_{P3} = 50$$

$$q_{P1} = 100 + 150 + 50 = 300$$

Step 2. Resistances. Scale the 1000 ft, 12 in value

$$R = 1.144 \times 10^{-5} \frac{L}{1000} \left(\frac{12}{D} \right)^{4.871}$$

$$R_{P1} = 2.47 \times 10^{-4} \quad R_{P2} = 6.70 \times 10^{-4}$$

$$R_{P3} = 3.35 \times 10^{-4}$$

Branched network heads, pressures, and a demand surge

Step 3. Walk downstream. With $\Delta h = Rq^{1.852}$, $h_{J1} = 900 - 9.57$, then J2 and J3 each subtract their own pipe's loss from h_{J1} , and finally $p = (h - z)/2.31$.

Pipe	q (GPM)	Δh (ft)	Node	h (ft)	p (psi)
P1	300	9.57	J1	890.43	47.8
P2	150	7.18	J2	883.25	40.4
P3	50	0.47	J3	889.96	56.3

Every node meets 40 psi, but J2 only just.

What if d_2 doubles to 300 GPM?

- Losses rise, P1 $9.57 \rightarrow 20.28$ ft and P2 $7.18 \rightarrow 25.91$ ft ($2^{1.852} = 3.6$ times)
- $h_{J2} = 853.82$ ft, so $p_{J2} = 27.6$ psi, **below 40 psi**
- Possible fixes are an 8 in P2, a booster pump, or a loop that feeds J2 from a second side

In a tree, mass fixes the flows and energy then gives the heads, so no Newton is needed. Loops are what make it hard.

Example of state estimation with least squares

Return to the branched network. Three flow meters read the pipe flows with noise

$$y = \begin{bmatrix} y_{P1} \\ y_{P2} \\ y_{P3} \end{bmatrix} = \begin{bmatrix} 301 \\ 148 \\ 49 \end{bmatrix} \text{ GPM}$$

and the readings break mass balance at J1, since $301 - 148 - 49 \neq 100$.

Use mass balance to write

$q_{P1} = 100 + q_{P2} + q_{P3}$, so with $x = [q_{P2} \ q_{P3}]^T$

$$y = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_H x + \underbrace{\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}}_{y_0} + \text{noise}$$

Least squares (a QP without constraints, Module 3)

$$\hat{x} = \arg \min_x \|y - y_0 - Hx\|^2$$

$$\hat{x} = (H^T H)^{-1} H^T (y - y_0) = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 349 \\ 250 \end{bmatrix}$$

$$\hat{q} = (299.67, 149.33, 50.33) \text{ GPM}$$

The estimate obeys mass balance exactly and sits within 0.7 GPM of the true flows (300, 150, 50).

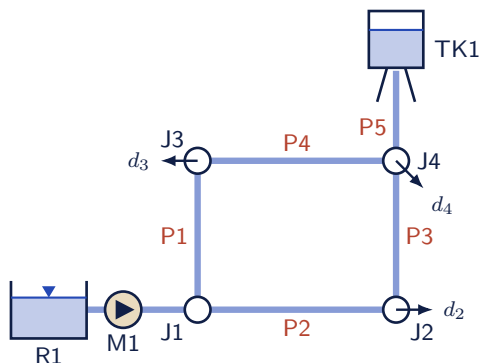
With pressure sensors the model is nonlinear, so the same least squares is solved by linearizing and iterating (Gauss-Newton).

PART 4

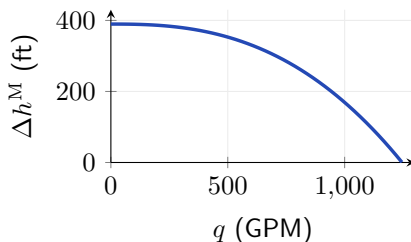
Solving the network

A looped network step by step, then Newton's method, a day of operation, and the state space view

Worked example with a looped network



pump curve, $s = 1$



Data

- Reservoir R1 has fixed head $h_{R1} = 700$ ft
- Pump M1 is fully on ($s = 1$) with $h_0 = 389.72$ ft, $r = 3.746 \times 10^{-6}$, $\nu = 2.59$, and it pumps into junction J1
- Demands are $d_2 = 200$, $d_3 = 100$, $d_4 = 150$ GPM, and J1 has none
- Every pipe has $R = 1.144 \times 10^{-5}$ (1000 ft, 12 in, $C = 100$) and $\mu = 1.852$
- Tank TK1 has $h_{TK1}(0) = 908$ ft and $\beta = 6.8 \times 10^{-5}$ ft/gal (a 50 ft diameter tank)
- One time step is one hour, $\Delta t = 60$ min

Operator task

Compute every head and flow now, and the head in tank TK1 after one hour.

During the hour the tank head is known (908 ft), so this hour's unknowns are 6 flows and 4 junction heads.

Step 1. Assume directions, then balance mass

Assume a direction for every link and write all equations with it. A negative answer means the water actually flows the other way. Any consistent choice works, but mixing conventions does not.

Mass balance at each junction (in – out = demand) reads

$$J1 \quad q_{M1} - q_{P1} - q_{P2} = 0$$

$$J2 \quad q_{P2} - q_{P3} = d_2 = 200$$

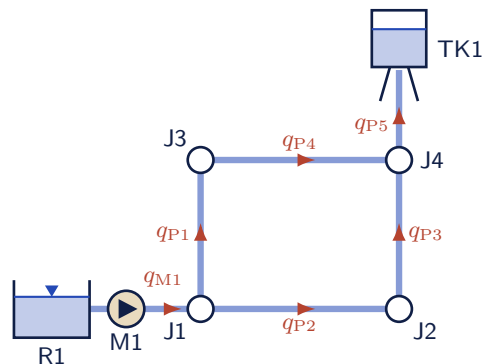
$$J3 \quad q_{P1} - q_{P4} = d_3 = 100$$

$$J4 \quad q_{P3} + q_{P4} - q_{P5} = d_4 = 150$$

Adding the four rows cancels every internal pipe

$$q_{M1} - q_{P5} = 450$$

The pump supplies the total demand plus whatever flows into the tank.



In matrix form this is $M_J q + d = 0$ with the incidence block of the previous section. Four linear equations cannot fix six flows, so energy must decide how the water splits around the loop.

Step 2. Energy balances and the tank update

Energy balance through the pump (head after = head before + gain) gives

$$h_{J1} - h_{R1} = h_{J1} - 700 = 389.72 - 3.746 \times 10^{-6} q_{M1}^{2.59}$$

Energy balance through each pipe (head drop = friction loss) gives

$$P1 \quad h_{J1} - h_{J3} = 1.144 \times 10^{-5} q_{P1} |q_{P1}|^{0.852}$$

$$P2 \quad h_{J1} - h_{J2} = 1.144 \times 10^{-5} q_{P2} |q_{P2}|^{0.852}$$

$$P3 \quad h_{J2} - h_{J4} = 1.144 \times 10^{-5} q_{P3} |q_{P3}|^{0.852}$$

$$P4 \quad h_{J3} - h_{J4} = 1.144 \times 10^{-5} q_{P4} |q_{P4}|^{0.852}$$

$$P5 \quad h_{J4} - h_{TK1} = h_{J4} - 908 = 1.144 \times 10^{-5} q_{P5} |q_{P5}|^{0.852}$$

Mass balance in the tank after one hour ($\Delta t = 60$ min, q in GPM) gives

$$h_{TK1}(1) = 908 + 6.8 \times 10^{-5} \times 60 \times q_{P5}$$

- 4 mass + 6 energy = 10 equations in 10 unknowns
- 6 of the 10 equations are nonlinear
- Solve the algebraic system first, then update the tank. This is the structure of the whole hydraulic model

Solution of the example

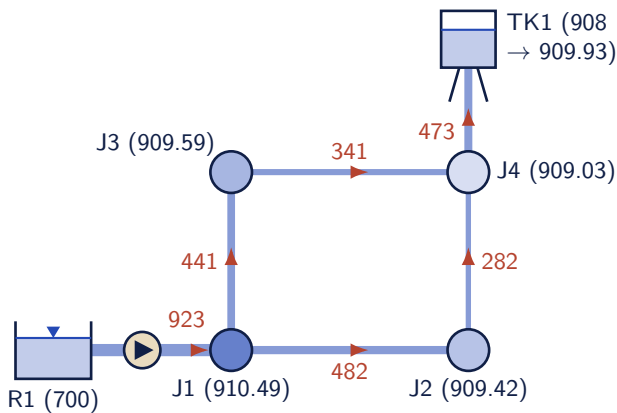
Solving the 10 equations gives

Flow	GPM	Head	ft
q_{M1}	922.97	h_{J1}	910.49
q_{P1}	440.64	h_{J2}	909.42
q_{P2}	482.34	h_{J3}	909.59
q_{P3}	282.34	h_{J4}	909.03
q_{P4}	340.64		
q_{P5}	472.97		

$$h_{TK1}(1) = 908 + 0.00408 \times 472.97$$

$$= \boxed{909.93 \text{ ft}}$$

- All flows are positive, so every assumed direction was right
- The pump adds 210.49 ft, and the pipes lose only about 1 ft each



Red labels are flows (GPM), and line width grows with flow. A darker node means a higher head (ft).

Checking the answer

Never trust a solver without checks.

① **Mass at J4.**

$$282.34 + 340.64 - 472.97 = 150.01 \approx d_4$$

② **Global mass.** $q_{M1} = 922.97 = 450 + 472.97$

③ **Energy around the loop**

$J1 \rightarrow J2 \rightarrow J4 \rightarrow J3 \rightarrow J1$ must sum to zero

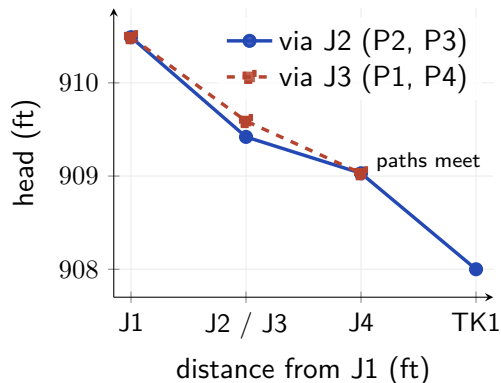
$$\underbrace{1.0666}_{P2} + \underbrace{0.3956}_{P3} - \underbrace{0.5601}_{P4} - \underbrace{0.9021}_{P1} = 0.0000$$

④ **Direction.** Water runs downhill in head on every pipe

⑤ **Tank.** $472.97 \times 60 = 28,378$ gal raise the level 1.93 ft

⑥ **Pressure** at J4 (if $z = 800$ ft) is 47.2 psi, inside 40 to 80 psi

Check 3 is Kirchhoff's voltage law for water, where heads play the role of potentials.



Hydraulic grade along the two paths from J1 to J4. Both paths must lose the same 1.46 ft, and that requirement decides how the flow splits (482 versus 441 GPM).

The worked example in Python

The ten equations go straight into a nonlinear solver, and the output matches the solution slide.

```
import numpy as np
from scipy.optimize import fsolve
R, mu = 1.144e-5, 1.852                # pipe resistance, HW exponent
h0, r, nu = 389.72, 3.746e-6, 2.59     # pump curve (s = 1)
hR, hT, d = 700.0, 908.0, (200, 100, 150)
hl = lambda q: R*q*abs(q)**(mu - 1)    # signed head loss
def F(z):
    qM, q1, q2, q3, q4, q5, h1, h2, h3, h4 = z
    return [qM-q1-q2, q2-q3-d[0], q1-q4-d[1], q3+q4-q5-d[2], # mass
            h1 - hR - (h0 - r*qM**nu),                        # pump
            h1-h3-hl(q1), h1-h2-hl(q2), h2-h4-hl(q3),        # pipes
            h3-h4-hl(q4), h4-hT-hl(q5)]
z = fsolve(F, [900, 450, 450, 250, 350, 450, 910, 909, 909, 909])
print(np.round(z, 2))                # flows (GPM), heads (ft)
print(round(hT + 6.8e-5*60*z[5], 2)) # tank head after 1 h is 909.93
```

fsolve uses a Newton type method, and the next slides explain why that works and when it struggles.

The water flow problem

General form. Given demands, fixed heads h_0 (reservoirs, tanks), and pump settings, find junction heads h_J and flows q with

$$M_J q + d = 0$$

$$M_J^T h_J + M_0^T h_0 = \Phi(q)$$

Φ_ℓ is the head loss of link ℓ ($-\Delta h^M$ for a pump).

As optimization, $\min_{q,h} 0$ subject to these equations is a *feasibility* problem.

- **Convex?** Not as written, since nonlinear equalities define a nonconvex set (Module 3)
- **Newton?** Yes, with the saddle point Jacobian

$$\begin{bmatrix} -\text{diag}(\Phi'(q)) & M_J^T \\ M_J & 0 \end{bmatrix}$$

- EPANET's global gradient algorithm (Todini and Pilati, 1988) is this Newton method
- Zero flow gives $\Phi' = 0$, so solvers floor the slope to keep the Jacobian invertible

A convex view of the water flow problem

Content minimization (fixed pump and valve status)

$$\min_q \sum_{\ell} \int_0^{q_{\ell}} \Phi_{\ell}(\sigma) d\sigma - h_0^T M_0 q$$

$$\text{s.t. } M_J q + d = 0$$

Each Φ_{ℓ} increases, so each integral is convex.

KKT stationarity with multiplier λ gives

$$\Phi(q) = M_0^T h_0 - M_J^T \lambda$$

This is exactly the energy balance with $h_J = -\lambda$.

The same equations are nonconvex as constraints but convex as optimality conditions.

- The water flow problem has a unique solution
- **Junction heads are the multipliers** of the mass balances, that is, the price of delivering one more GPM (Module 3 duality)
- Valve and pump switching break this convexity

Newton's method on the operating point

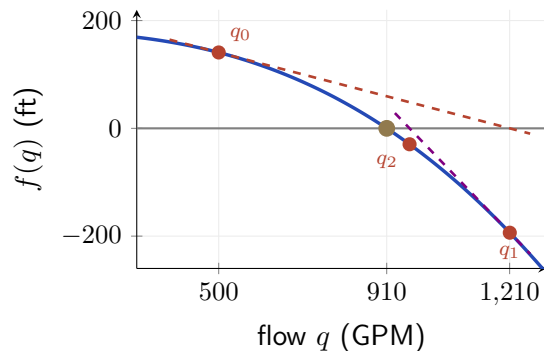
Return to the single pump and pipe example
with $s = 1$

$$f(q) = h_0 - r q^\nu - 210 - R q^\mu = 0$$

$$f'(q) = -r\nu q^{\nu-1} - R\mu q^{\mu-1}$$

$$q_{t+1} = q_t - \frac{f(q_t)}{f'(q_t)}$$

t	q_t (GPM)	$f(q_t)$ (ft)
0	500.00	140.80
1	1210.36	-193.67
2	965.78	-29.56
3	912.56	-1.27
4	910.07	-2.7×10^{-3}
5	910.06	-1.2×10^{-8}



- The error falls $2.5 \rightarrow 0.005 \rightarrow 10^{-8}$, which is quadratic convergence
- The first step overshoots because f curves strongly, and a damped Newton step with backtracking (Module 3, Algorithm 2) would shorten it
- The network version does the same with a 10×10 Jacobian

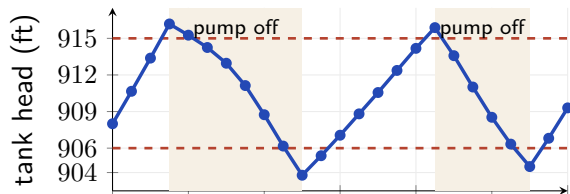
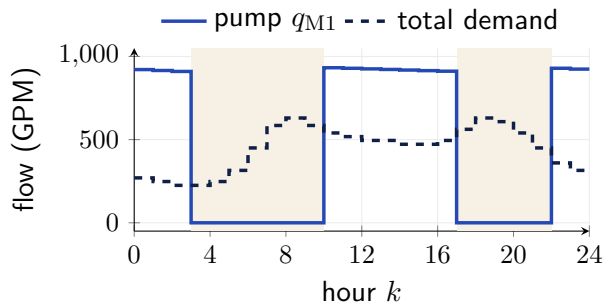
A day of operation with extended period simulation

Repeat the hourly solve for 24 hours

- ① Demands follow a daily pattern
 $d_i(k) = m_k \bar{d}_i$, $m_k \in [0.5, 1.4]$
- ② Rule based control turns the pump *off* if $h_{TK1} \geq 915$ ft, *on* if $h_{TK1} \leq 906$ ft, checked each hour
- ③ Solve the water flow problem with the tank head fixed
- ④ Update the tank with
 $h(k+1) = h(k) + \beta \Delta t q_{P5}(k)$

What happens

- The pump runs 12 of 24 hours
- Hourly checks overshoot the band, reaching 916.2 and 903.8 ft
- **Flows reverse** when the pump stops, so the tank feeds the town and P1, P3, P4, P5 all change sign



Hydraulics in state space as a nonlinear DAE

Three groups of equations

$$h^{\text{TK}}(k+1) = h^{\text{TK}}(k) + E^{\text{TK}} q(k)$$

$$0 = M_J q(k) + d(k)$$

$$0 = M^T h(k) - \Phi(q(k))$$

E^{TK} holds $\beta\Delta t$ times the tank rows of the incidence matrix.

Stacking $x = \{h, q\}$ gives

$$E x(k+1) = A_x x(k) + E_d d(k) + E_\Phi \Phi(x(k))$$

Is E singular? Yes, since only the tank rows have a $(k+1)$ term

$$E = \begin{bmatrix} I_{n_{\text{TK}}} & 0 \\ 0 & 0 \end{bmatrix}$$

In the example $x \in \mathbb{R}^{11}$ (5 heads, 6 flows) and $E = \text{diag}(1, 0, \dots, 0)$.

- It is a **difference algebraic equation** (DAE) with one dynamic state and ten algebraic unknowns
- There is no explicit $x(k+1) = f(x(k))$, so each step solves a water flow problem
- The model is quasi steady, so pressure waves (water hammer) are ignored

Linearized hydraulics

Linearize around an operating point $\bar{q}$ (Module 4)

$$\Phi_\ell(q) \approx \Phi_\ell(\bar{q}) + \Phi'_\ell(\bar{q})(q - \bar{q})$$

Pipe P5 at $\bar{q} = 472.97$ GPM gives

$$\Phi' = \mu R |\bar{q}|^{\mu-1} = 4.03 \times 10^{-3} \frac{\text{ft}}{\text{GPM}}$$

Ten more GPM cost 0.040 ft of extra loss.

Pump M1 at $\bar{q} = 922.97$ GPM gives

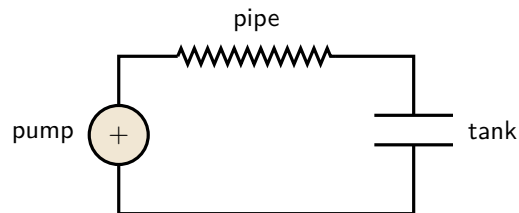
$$\frac{d\Delta h^M}{dq} = -r\nu \bar{q}^{\nu-1} = -0.503 \frac{\text{ft}}{\text{GPM}}$$

The pump is 125 times stiffer than the pipe, so the pump curve, not the pipe, sets the flow.

- The local model $M_J \delta q = -\delta d$, $M^T \delta h = \text{diag}(\Phi') \delta q$ is a network of resistors
- It fails near $q = 0$ (zero slope) and when pumps switch

The circuit analogy

Water network	Electric circuit
head h	voltage
flow q	current
pipe	nonlinear resistor
tank	capacitor ($\beta \leftrightarrow 1/C$)
reservoir	ideal voltage source
pump	nonlinear source
check valve	diode
mass balance	Kirchhoff current law
loop energy	Kirchhoff voltage law



A pump filling a tank through a pipe, drawn as a source charging a capacitor through a resistor.

Module 4's RLC circuit and a water network are the same kind of system.

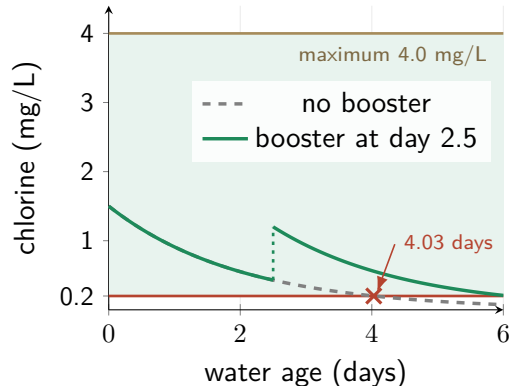
PART 5

Water quality models

How chlorine mixes, decays, and travels, ending in a linear state space model

Why chlorine, and why it is hard to keep

- Chlorine inactivates pathogens, and the **residual** left in the pipes guards against later contamination (main breaks, intrusion)
- US rules require at least 0.2 mg/L entering the distribution system and a detectable residual throughout (Surface Water Treatment Rule), and at most 4.0 mg/L (maximum residual disinfectant level)
- Too little brings microbial risk and boil water notices, and too much brings disinfection byproducts and taste and odor complaints
- Chlorine **decays** by reacting with the water (bulk) and pipe walls (wall), so old water loses it and boosters re-inject it
- It is a good quality proxy, cheap to measure online, and a sudden drop flags contamination



Water leaving the plant at 1.5 mg/L with bulk decay $K_b = 0.5$ per day drops below 0.2 mg/L after 4.03 days of travel and storage. A booster restores it.

Chlorine mass balance

Mass rate. Concentration c (mg/L) times flow q (GPM) is a mass rate

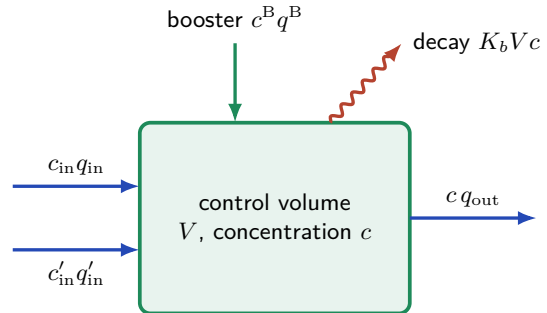
$$3.785 \, c \, q \quad \text{mg/min}$$

For example, 1.2 mg/L at 500 GPM carries 2271 mg/min, or 3.27 kg/day.

Conservation of mass for any control volume reads

$$\frac{d(Vc)}{dt} = \sum_{\text{in}} cq - \sum_{\text{out}} cq - K_b V c + \text{injection}$$

Every quality model in this section is this one balance applied to a junction, a tank, or a pipe.



First order decay

With no flow in or out, $\dot{c} = -K_b c$, so

$$c(t) = c_0 e^{-K_b t}$$

K_b is of order 0.1 to 1 per day, and pipe walls add more decay.

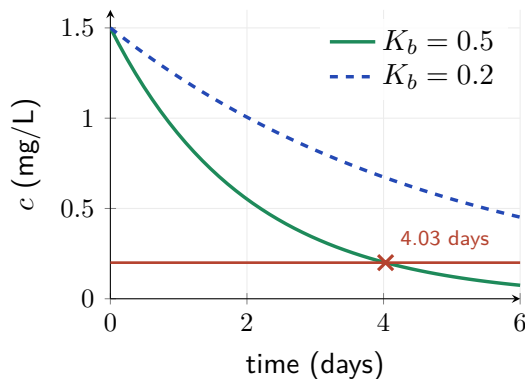
Example. From 1.5 to 0.2 mg/L at $K_b = 0.5$ per day takes

$$t = \frac{\ln(1.5/0.2)}{0.5} = 4.03 \text{ days}$$

Over one time step

$$c(k+1) = e^{-K_b \Delta t} c(k) \approx (1 - \kappa) c(k)$$

with $\kappa = K_b \Delta t$, exact versus Euler as in Module 4.



The red line is the 0.2 mg/L minimum.

Instantaneous mixing at a junction

- A junction stores nothing, so mixing is complete and instantaneous
- Every stream leaving the junction, demand included, carries the same mixed concentration c_1^J
- Consider dilution only (no booster, no decay)

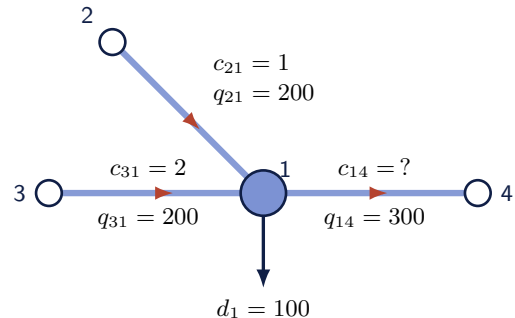
Mass in = mass out gives

$$c_{21}q_{21} + c_{31}q_{31} = c_1^J q_{14} + c_1^J d_1$$

$$c_1^J = \frac{c_{21}q_{21} + c_{31}q_{31}}{d_1 + q_{14}} = \frac{1 \times 200 + 2 \times 200}{100 + 300}$$

$$= \boxed{1.25 \text{ mg/L}}$$

The result is a flow weighted average, so it lies between the smallest and largest incoming concentration.



Concentrations in mg/L, flows in GPM.

Hydraulics first

Every weight in this average is a flow. The quality model cannot be written until the water flow problem is solved.

The general junction model with a booster

With L_{in} the links flowing into junction i and L_{out} the links leaving it,

$$c_i^J = \frac{\sum_{ki \in L_{\text{in}}} c_{ki} q_{ki}}{d_i + \sum_{ij \in L_{\text{out}}} q_{ij}} = \sum_{ki \in L_{\text{in}}} w_{ki} c_{ki}$$

With a booster that injects flow q^B at concentration c^B ,

$$c_i^J = \sum_{ki \in L_{\text{in}}} w_{ki} c_{ki} + w^B c^B$$

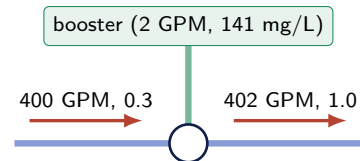
- Weights $w = q/(\text{total outflow})$ are nonnegative and sum to one, a convex combination
- c_i^J is **linear** in the incoming concentrations and in c^B , and the flows only set the weights

Example of sizing a booster. A junction receives 400 GPM at 0.3 mg/L. Customers downstream need 1.0 mg/L. The booster pumps 2 GPM of concentrate, so

$$1.0 = \frac{0.3 \times 400 + c^B \times 2}{402}$$

$$c^B = \frac{402 - 120}{2} = \boxed{141 \text{ mg/L}}$$

The chlorine dose is $3.785 \times 141 \times 2 = 1067$ mg/min, about **1.54 kg/day**.



Complete mixing in a tank

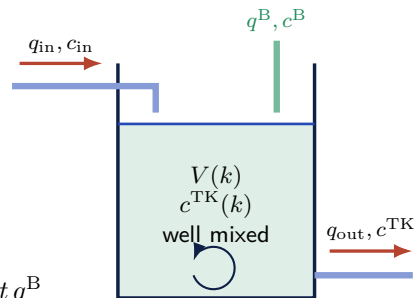
The mass balance over one step is

$$V(k+1)c^{\text{TK}}(k+1) = V(k)c^{\text{TK}}(k)(1 - \kappa) - \Delta t q_{\text{out}}c^{\text{TK}}(k) + \Delta t q_{\text{in}}c_{\text{in}}(k) + \Delta t q^{\text{B}}c^{\text{B}}(k)$$

with $V(k+1) = V(k) + \Delta t (q_{\text{in}} + q^{\text{B}} - q_{\text{out}})$. Dividing,

$$c^{\text{TK}}(k+1) = a^{\text{TK}}(k)c^{\text{TK}}(k) + a_{\text{in}}(k)c_{\text{in}}(k) + b^{\text{TK}}(k)c^{\text{B}}(k)$$

$$a^{\text{TK}} = \frac{V(k)(1 - \kappa) - \Delta t q_{\text{out}}}{V(k+1)}, \quad a_{\text{in}} = \frac{\Delta t q_{\text{in}}}{V(k+1)}, \quad b^{\text{TK}} = \frac{\Delta t q^{\text{B}}}{V(k+1)}$$



- **Draining** does not dilute, since with $q_{\text{in}} = 0$ and no decay $a^{\text{TK}} = 1$
- The coefficients follow $V(k)$ and $q(k)$, so they are **time varying**

Tank example and water age

Data. $V = 100,000$ gal at $c^{\text{TK}} = 1.0$ mg/L. Over $\Delta t = 10$ min, inflow is 500 GPM at 1.4 mg/L, outflow is 300 GPM, and $K_b = 0.5$ per day, so $\kappa = 0.5 \times 10/1440 = 0.00347$.

Chlorine ledger	mg/L $\times$ gal
stored ($100,000 \times 1.0$)	100,000
decay ($-0.00347 \times 100,000$)	-347
out ($-300 \times 10 \times 1.0$)	-3,000
in ($+500 \times 10 \times 1.4$)	+7,000
end of step	103,653

$$V(k+1) = 102,000 \text{ gal}$$

$$c^{\text{TK}}(k+1) = \frac{103,653}{102,000} = \boxed{1.016 \text{ mg/L}}$$

Without decay the answer would be $104,000/102,000 = 1.020$ mg/L.

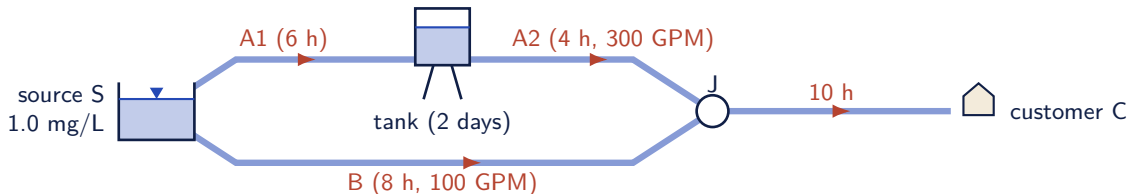
Water age is the time water has spent in the network. Tanks set most of it.

- A tank that cycles only a small fraction of its volume each day holds water for days
- At $K_b = 0.5$ per day, three days of age remove $1 - e^{-1.5} = 78\%$ of the chlorine
- Operators deliberately draw tanks down to turn the water over

Tank mixing models (EPANET)

- Complete mix (used here)
- Two compartment
- Plug flow, first in first out
- Stacked plug flow, last in first out

Does the farthest customer get chlorine?



Model. Each pipe and the tank is a delay with decay

$$c_{\text{out}} = c_{\text{in}} e^{-K_b \text{ age}}, \quad K_b = 0.5 \text{ per day}$$

J mixes the two paths by flow.

- Path A takes 58 h (2.42 d), so
 $c_A = 0.299$
- Path B takes 8 h (0.33 d), so
 $c_B = 0.846$

Junction J mixes

$$c^J = \frac{300 \times 0.299 + 100 \times 0.846}{400} = 0.436$$

Customer C, 10 h later, gets

$$0.436 e^{-0.5 \times 10/24} = \boxed{0.354 \text{ mg/L}}$$

Above the 0.2 mg/L minimum.

Example continued with a tank that holds water too long

Now the tank holds water for 4 days instead of 2. Path A ages 106 h (4.42 days).

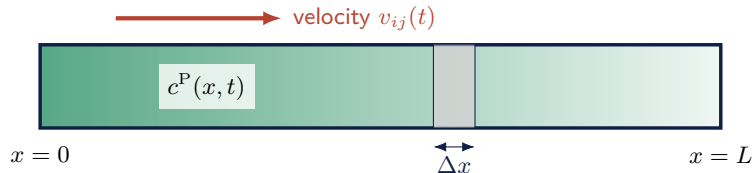
(mg/L)	tank 2 days	tank 4 days
after tank, c_A	0.299	0.110
junction J	0.436	0.294
customer C	0.354	0.239

- C still passes, but customers right after the tank get **0.110 mg/L**
- Blending with fresh water at J hides the problem from a sensor at J

Three fixes

- ① Raise the source dose. The model is linear, so $c_0 \geq 0.2 e^{0.5 \times 4.42} = 1.82$ mg/L, which overdoses customers near S
- ② Add a booster at the tank outlet, so only path A water gets more chlorine
- ③ Cycle the tank harder to cut its age back toward 2 days

The advection-reaction equation in a pipe



$$\frac{\partial c^P}{\partial t} = \underbrace{-v_{ij}(t) \frac{\partial c^P}{\partial x}}_{\text{advection (transport)}} \underbrace{- K_b c^P}_{\text{reaction (decay)}}$$

Constant velocity has an exact answer. Along characteristics $x = vt$,

$$c^P(x, t) = c^P\left(0, t - \frac{x}{v}\right) e^{-K_b x/v}$$

The pipe is a pure delay of x/v plus attenuation.

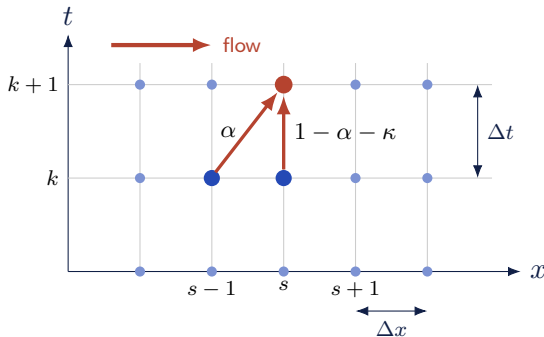
Example. P1 carries 440.64 GPM in a 12 in pipe, so $v = q/A = 0.982/0.785 = 1.25$ ft/s. Over 1000 ft the delay is 800 s (13.3 min) and decay removes only 0.5%.

Why discretize anyway?

- $v_{ij}(t)$ changes with demand every hour
- Flows reverse, so inlet and outlet swap
- Thousands of pipes couple through junction mixing
- Reactions can be nonlinear ($K c^n$) or involve several species

The **objective** is to solve this PDE in every pipe, and the **approach** is to discretize in space and time.

Upwind discretization



The new value at segment s uses only itself and its *upstream* neighbor.

Replace the derivatives by one sided differences that look upstream

$$\frac{\partial c}{\partial t} \approx \frac{c(s, k+1) - c(s, k)}{\Delta t}$$

$$\frac{\partial c}{\partial x} \approx \frac{c(s, k) - c(s-1, k)}{\Delta x}$$

$$c(s, k+1) = \alpha c(s-1, k) + (1 - \alpha - \kappa) c(s, k)$$

$$\alpha = \frac{v \Delta t}{\Delta x}, \quad \kappa = K_b \Delta t$$

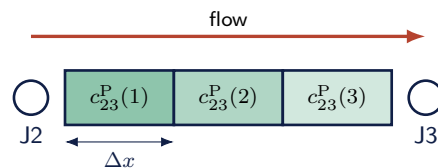
- **Stability (CFL condition)** needs $0 \leq \alpha$ and $\alpha + \kappa \leq 1$. The update is then a weighted average, so it cannot overshoot. In practice $\Delta t \leq \Delta x/v$ for the fastest pipe, which forces short time steps
- $\alpha = 1$, $\kappa = 0$ shifts the profile exactly one segment per step, while $\alpha < 1$ smears fronts (numerical diffusion)
- If the flow reverses, *upstream* becomes $s+1$, so the stencil follows the sign of v

A pipe with three segments, in matrix form

Split pipe 23 (from J2 to J3) into three segments and apply the upwind update to each, $s = 1, 2, 3$,

$$c_{23}^P(s, k+1) = \alpha c_{23}^P(s-1, k) + (1-\alpha-\kappa) c_{23}^P(s, k)$$

Segment 0 is junction J2, so $c_{23}^P(0, k) = c_2^J(k)$.



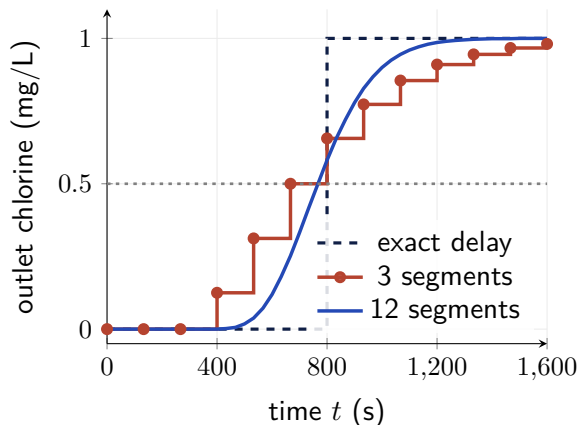
$$\underbrace{\begin{bmatrix} c_{23}^P(1, k+1) \\ c_{23}^P(2, k+1) \\ c_{23}^P(3, k+1) \end{bmatrix}}_{c_{23}^P(k+1)} = \underbrace{\begin{bmatrix} 1-\alpha-\kappa & 0 & 0 \\ \alpha & 1-\alpha-\kappa & 0 \\ 0 & \alpha & 1-\alpha-\kappa \end{bmatrix}}_{A_{23}} \underbrace{\begin{bmatrix} c_{23}^P(1, k) \\ c_{23}^P(2, k) \\ c_{23}^P(3, k) \end{bmatrix}}_{c_{23}^P(k)} + \underbrace{\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}}_{b_{23}} c_2^J(k)$$

- It is lower triangular, so every eigenvalue is $1-\alpha-\kappa \in [0, 1)$ and the pipe is **stable** (Module 4)
- (A_{23}, b_{23}) is controllable, so the inlet can steer every segment

Upwind example with a chlorine front traveling down P1

P1 is 1000 ft long with $v = 1.25$ ft/s. At $t = 0$ the inlet steps from 0 to 1 mg/L. Decay over 13 minutes is negligible, so $\kappa \approx 0$.

Model	Δx	Δt	α
3 segments	333 ft	133 s	0.5
12 segments	83 ft	33 s	0.5
exact			delay 800 s



Finer segments sharpen the front but multiply the number of states and shorten the time step.

- Upwind lets chlorine arrive early and smears the front, which is **numerical diffusion**
- Half the step arrives at 667 s (3 segments) or 767 s (12) instead of 800 s
- $\alpha = 1$ is exact, but one Δt cannot match $\Delta x/v$ in every pipe, so EPANET tracks moving water parcels instead (Lagrangian)

All components in one table

Pumps and valves store nothing and add no delay, so $c_{ij}^M(k) = c_i(k)$.

Reservoirs hold a fixed concentration, so $c^R(k+1) = c^R(k)$.

Component	Water quality model	Meaning
Junction	$c^J(k) = A_J^L(k) c^L(k) + B_J(k) c^B(k)$	mixing
Tank	$c^{\text{TK}}(k+1) = A_{\text{TK}}^{\text{TK}}(k) c^{\text{TK}}(k) + A_{\text{TK}}^{\text{P}}(k) c^{\text{P}}(k) + B_{\text{TK}}(k) c^{\text{B}}(k)$	mixing
Pipe	$c^{\text{P}}(k+1) = A_{\text{P}}^{\text{P}}(k) c^{\text{P}}(k) + A_{\text{N}}^{\text{P}}(k) c^{\text{N}}(k)$	upwind
Pump, valve	$c^{\text{M}}(k) = E^{\text{NM}} c^{\text{N}}(k)$	copy

c^{N} stacks all node concentrations and c^{L} the link outlet concentrations (last pipe segments, pumps, valves).

Every coefficient matrix is built from the flows of the hydraulic solution.

The network quality model

Linear difference equation

Stack $\chi = \{c^J, c^R, c^{TK}, c^P, c^M, c^V\}$ and $u = c^B$, then substitute the algebraic junction equations to get

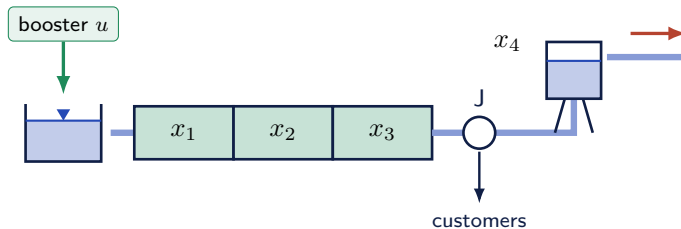
$$\chi(k+1) = A(k)\chi(k) + B(k)u(k), \quad y(k) = C\chi(k)$$

y holds the chlorine sensor readings.

- **Linear** in χ and u , so superposition holds
- **Time varying**, since $A(k), B(k)$ follow the hourly flows
- **Large and sparse**, with 10^4 to 10^6 states
- **Positive and stable**, with nonnegative entries and row sums at most one

This is the model of Module 4, so its stability, controllability, and observability tests apply directly.

Example of a four state quality model

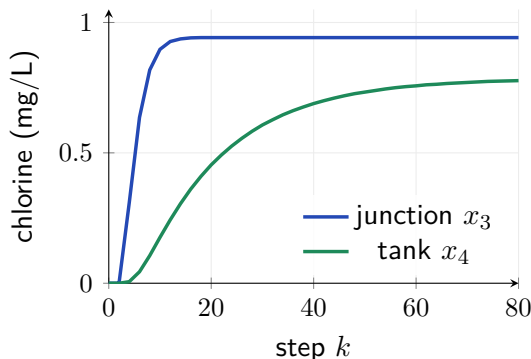


- x_1, x_2, x_3 are pipe segments with $\alpha = 0.5, \kappa = 0.01$
- Junction J has $c^J = x_3$
- x_4 is a flow through tank exchanging 5% of its volume per step, decay 0.01

$$A = \begin{bmatrix} 0.49 & 0 & 0 & 0 \\ 0.5 & 0.49 & 0 & 0 \\ 0 & 0.5 & 0.49 & 0 \\ 0 & 0 & 0.05 & 0.94 \end{bmatrix}$$

$$B = [0.5 \quad 0 \quad 0 \quad 0]^T$$

Four state model response and Module 4 tests



Constant booster setting $u = 1$ mg/L from a chlorine free start.

- Eigenvalues 0.49, 0.49, 0.49, 0.94 make it stable
- The tank is the slow mode, with a time constant of 16 steps
- Steady state gives $c^J = 0.942u$ and $c^{TK} = 0.785u$

Module 4 tests

- It is controllable from the booster, since $\text{rank}[B \ AB \ A^2B \ A^3B] = 4$
- A sensor in the tank, $C = [0 \ 0 \ 0 \ 1]$, makes it observable with rank 4
- A sensor at the junction, $C = [0 \ 0 \ 1 \ 0]$, gives rank 3. **It cannot see the tank downstream**

Simulating a real network in Python

For real networks, describe the network in an EPANET input file and let a library run the hydraulics and the chlorine model together over a day.

```
import wntr
wn = wntr.network.WaterNetworkModel('loop.inp') # EPANET input file
wn.options.time.duration = 24*3600             # one day, in seconds
wn.options.quality.parameter = 'CHEMICAL'       # track chlorine
res = wntr.sim.EpanetSimulator(wn).run_sim()
head, flow = res.node['head'], res.link['flowrate']
chlorine = res.node['quality']                  # time by node table
```

- EPANET solves the water flow problem at every hydraulic step with Newton's method (the global gradient algorithm)
- It then advances chlorine with the network quality model on a shorter time step
- WNTR wraps EPANET in Python and adds tools for resilience studies

PART 6

Operating the network

Pump scheduling, chlorine dosing, and where to place
boosters and sensors

The general pump scheduling problem

Choose pump status or speed for each hour of the day

$$\min_{s(\cdot), x(\cdot)} \sum_{k=0}^{N-1} \pi_k P_k(q^M(k), \Delta h^M(k)) \Delta t$$

subject to, for $k = 0, \dots, N - 1$,

- the hydraulic DAE (tank dynamics, mass, and energy balances)
- tank limits $h_{\min}^{\text{TK}} \leq h^{\text{TK}}(k) \leq h_{\max}^{\text{TK}}$
- service pressure $p_i(k) \geq p_{\min}$ at every junction
- end the day as full as it started,
 $h^{\text{TK}}(N) \geq h^{\text{TK}}(0)$
- $s_j(k) \in \{0, 1\}$ (fixed speed) or $s_j(k) \in [s_{\min}, 1]$ (variable speed)

π_k is the electricity price (\$/kWh) and P_k the pump power from the formula on the pump slide.

Storage is what makes scheduling worthwhile, since tanks let pumping happen when energy is cheap.

Problem class (Module 3)

Binary status and nonlinear energy balances make a **mixed integer nonlinear program**, nonconvex even with continuous speeds.

Practical approaches

- Linearize the hydraulics around the previous schedule and solve LPs or QPs in sequence
- Convex relaxations and geometric programming
- Model predictive control, re-solving every hour with new demand forecasts (Module 4)
- Heuristics (genetic algorithms) and rule based control in industry

Pump scheduling example as an LP

Keep only the tank. Treat the pump head as fixed at 215 ft with $\eta = 0.75$, so power is 0.054 kW per GPM.

$$\min_{q_0, \dots, q_{23}} \sum_{k=0}^{23} \pi_k \cdot 0.054 q_k$$

subject to

$$h_{k+1} = h_k + \beta \Delta t (q_k - d_k)$$

$$905 \leq h_k \leq 920, \quad 0 \leq q_k \leq 1000$$

$$h_0 = 910, \quad h_{24} \geq h_0$$

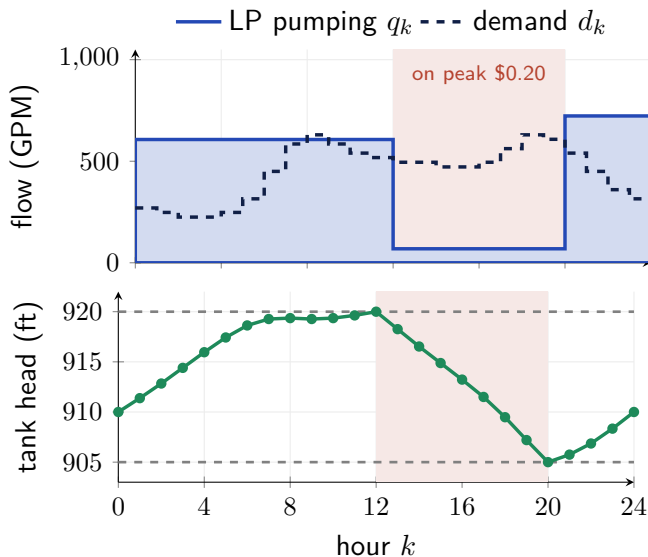
Data

- Price $\pi_k = \$0.08/\text{kWh}$ off peak, $\$0.20/\text{kWh}$ from noon to 8 pm
- Demand d_k follows the daily pattern of the 24 hour simulation
- The tank band is 905 to 920 ft

Problem class

Linear cost, linear dynamics, and box bounds make an LP in 24 variables (Module 3).

Pump scheduling example results



- **\$49.95 per day** versus \$73.78 when pumping follows demand
- **32% less** for the same 580 kWh
- The tank fills to 920 ft by noon and drains to 905 ft by 8 pm
- It is too small to skip the peak entirely

A tiny penalty on hour to hour changes picks the smoothest of several equally cheap schedules.

Chlorine dosing as an LP, and as MPC

With the flows known from hydraulics, choose booster settings over a horizon of N steps

$$\min_{u(0), \dots, u(N-1)} \sum_{k=0}^{N-1} q^B(k)^T u(k) \quad (\text{chlorine used})$$

subject to

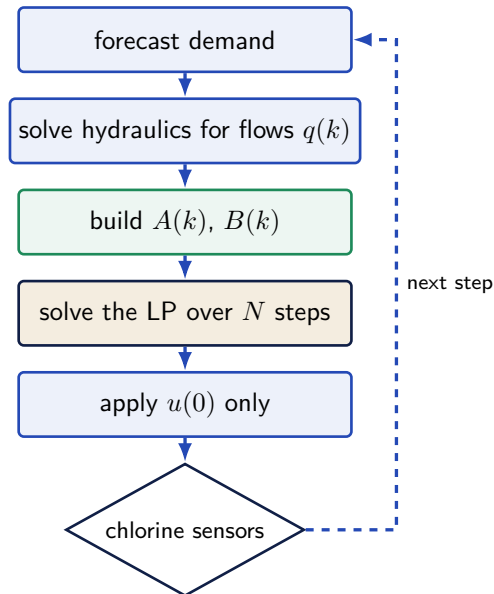
$$\chi(k+1) = A(k)\chi(k) + B(k)u(k)$$

$$\chi_{\min} \leq C\chi(k) \leq \chi_{\max} \quad (\text{monitored nodes})$$

$$u_{\min} \leq u(k) \leq u_{\max}$$

- Linear dynamics, linear cost, and linear bounds give an **LP**, and a smoothing penalty on u makes it a QP
- Stack the predictions $X = \mathcal{A}\chi(0) + \mathcal{B}U$ exactly as in the MPC section of Module 4
- If the flows are decisions too, $A(q)\chi$ is a product of unknowns and the problem is nonconvex again

Receding horizon loop



Dosing example as an LP

Use the four state model. It starts at the steady state of $u = 0.6$, where the tank has only 0.47 mg/L.

$$\min_{u(0), \dots, u(79)} \sum_{k=0}^{79} u(k)$$

subject to the model and

$$0.8 \leq x_4(k) \leq 2.0 \quad (k \geq 25)$$

$$0.3 \leq x_3(k) \leq 4.0$$

$$0 \leq u(k) \leq 4$$

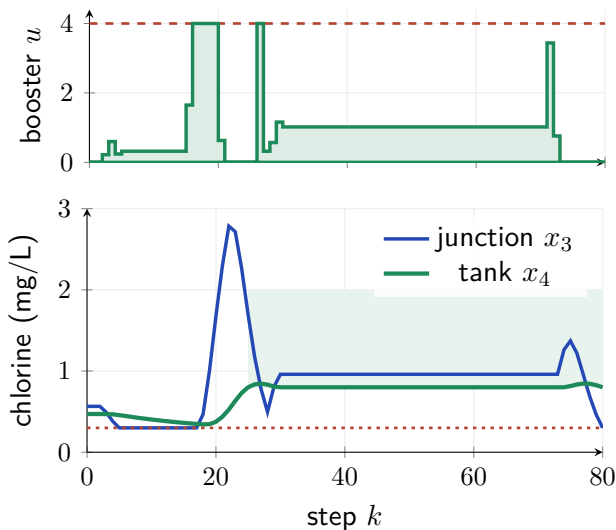
In words

- Bring the tank into its band by step 25 and keep it there
- Never let the junction drop below 0.3 mg/L or exceed the 4.0 mg/L maximum
- Use as little chlorine as possible

Baseline

The cheapest *constant* setting that meets the band is $u = 1.21$, with total dose 96.8.

Dosing example results



Total dose 74.6 versus 96.8, so
23% less chlorine.

- Doses at the limit for steps 16 to 19 so the slug reaches the tank by step 25
- Then holds
 $u = 1.02 = 0.8/0.785$, so the tank sits on its bound
- Stops near $k = 73$ because the horizon ends, and a receding horizon fixes this

Where to put boosters and sensors

Booster placement ($z_j = 1$ builds a booster at candidate node j , fixed cost f_j)

$$\min_{z,u} \sum_j f_j z_j + \sum_{k,j} u_j(k)$$

s.t. quality model and bounds, $0 \leq u_j(k) \leq u_{\max} z_j$

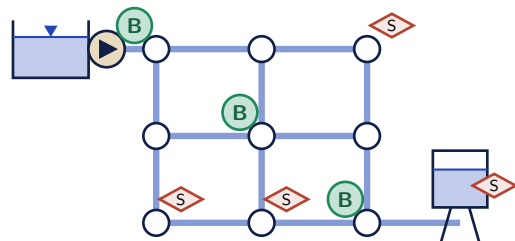
$$z_j \in \{0, 1\}$$

A mixed integer LP for fixed flows.

Sensor placement ($\sigma_j = 1$ installs a sensor at node j)

$$\max_{\sigma \in \{0,1\}^p} \log \det W_o(\sigma) \quad \text{s.t.} \quad \sum_j \sigma_j \leq n_s$$

or minimize the expected time to detect contamination (EPA's TEVA-SPOT tool).



B marks a candidate booster and S a candidate sensor.

Link to Module 4

- Boosters must reach every monitored node, a controllability question
- Sensors must reconstruct the residual, an observability question
- Rank says *whether* and Gramians say *how well*

Modern directions in water network operations

Digital twins

A calibrated hydraulic and quality model runs alongside the real network, fed in real time by SCADA and smart meters (AMI).

Estimation and leaks

Least squares state estimation from pressure and flow sensors (a QP, Module 3). A leak shows up as demand that the model cannot explain.

Forecasting

Machine learning forecasts of hourly demand feed MPC for pumps and boosters (the learning examples of Module 3).

Resilience

Pressure driven demand models for main breaks, power outages, fire flow, and freezes, when customers cannot receive their full demand.

Water-energy nexus

Pumps with storage are flexible electrical loads that can shift consumption to help the power grid (demand response).

Open tools

EPANET 2.2 (US EPA) for hydraulics and quality, EPANET-MSX for multiple species, and WNTR (Python, EPA and Sandia) for resilience studies.

The models of this module are the core of every one of these tools.

Module recap

Network and hydraulics

- Nodes carry heads and links carry flows, and the incidence matrix M writes every balance
- Mass balance is linear, $M_J q + d = 0$
- Energy balance is nonlinear, $M^T h = \Phi(q)$, with $Rq|q|^{0.852}$ for pipes and $s^2(h_0 - r(q/s)^\nu)$ for pumps
- Newton's method solves the water flow problem, and with fixed statuses it is the KKT system of a convex problem whose multipliers are the heads
- Over time the model is a DAE, so solve the network and then update the tanks

Water quality

- Junctions mix by flow weighted averages and tanks mix by volume
- Pipes obey an advection-reaction PDE, and upwind gives $c(s, k+1) = \alpha c(s-1, k) + (1-\alpha-\kappa)c(s, k)$, stable when $\alpha + \kappa \leq 1$
- The network model $\chi(k+1) = A(k)\chi(k) + B(k)u(k)$ is linear, time varying, positive, and stable

Operations

- Pump scheduling uses storage to shift energy use, and it is an MINLP that becomes an LP after approximation
- Chlorine dosing is an LP or MPC for given flows
- Booster and sensor placement are controllability and observability problems with binary choices