

Module 06

Transportation Systems and Traffic Flow

Ahmad F. Taha

**CE 4240/5240 — Introduction to
Infrastructure Systems Engineering**

Email ahmad.taha@vanderbilt.edu

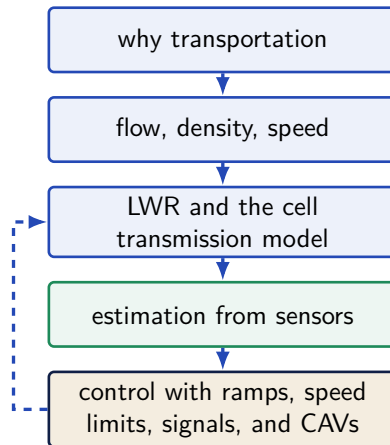
Webpage <https://ahmadstatespace.github.io/>

Fall 2026



Outline and objectives

- ① Explain why traffic is a dynamic system worth controlling
- ② Relate flow, density, and speed through the fundamental diagram
- ③ Derive the LWR conservation law, then compute shock waves and queues by hand
- ④ Simulate a freeway with the cell transmission model and write it in state space
- ⑤ Estimate traffic densities from a few sensors and decide where to place them
- ⑥ Design ramp metering, speed limits, and signal timing as feedback and optimization problems



Module 3 supplies least squares and LPs, Module 4 supplies observability and feedback, and Module 5 supplies conservation laws and upwind schemes.

Notation

Traffic flow (per lane)

Symbol	Meaning
x, t	position (mi), time (h)
ρ	density (veh/mi)
Q	flow (veh/h)
v	speed (mph)
v_f	free flow speed
w	backward wave speed
ρ_c, ρ_m	critical and jam density
$Q_{\max}$	capacity, $Q_{\max} = v_f \rho_c$
$Q_e(\rho)$	fundamental diagram
s	shock speed

Models and control

Symbol	Meaning
i, k	cell index, time step
l, T	cell length (mi), step (h)
δ_i, σ_i	demand and supply of cell i
q_i	flow from cell i to $i + 1$
$\rho(k)$	state, the vector of cell densities
r	on-ramp flow, the metering rate
y	measurements
C, W_o	output matrix, observability Gramian
K	feedback gain

The letter x always means position. The state of a discretized road is $\rho(k)$, not x .

PART 1

Why transportation

Emissions, safety, and congestion, and why traffic is a dynamic system worth modeling and controlling

Why transportation systems

Emissions

Transportation is the largest source of US greenhouse gas emissions, about 28% of the total, and almost all of it is CO₂ from burning fuel.

Safety

About 40,000 people die on US roads every year. Vision Zero, started in Sweden in 1997, aims for a road system with no deaths or serious injuries.

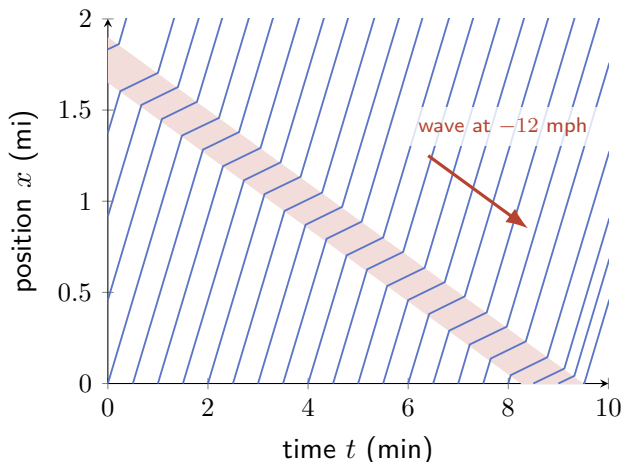
Congestion

Jams appear every day, often with no crash or lane closure at all. Adding a road can even make everyone slower (Braess's paradox).

- Traffic is a **dynamic system** with memory, since vehicles on a road are stored like water in a pipe
- It is nonlinear, large scale, and driven by people
- Operators have real actuators, such as ramp meters, speed limits, and signals
- And real sensors, such as loop detectors, cameras, and phones

Sources are the US EPA greenhouse gas inventory and NHTSA fatality reports.

Stop-and-go waves in space and time

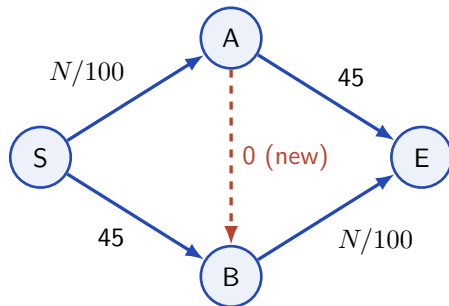


Each line is one vehicle. Lines are steep at 55 mph and flat inside the slow band.

- A **space-time diagram** plots vehicle trajectories $x_n(t)$, and the slope of a line is the vehicle's speed
- Vehicles move forward, but the slow band moves *backward*
- Such phantom jams start from small braking events, with no crash at all
- The I-24 MOTION testbed near Nashville tracks every vehicle over about four miles and sees these waves daily, moving back at about 12 mph

Information in congested traffic travels against the flow of vehicles.

Congestion is a network problem, Braess's paradox



Travel times in minutes. N is the number of drivers on that link.

4000 drivers travel from S to E, and each picks the fastest route for themselves.

Without the new link they split evenly

$$\frac{2000}{100} + 45 = 65 \text{ min for everyone}$$

With a free link from A to B the route S A B E always looks best, so everyone takes it

$$\frac{4000}{100} + 0 + \frac{4000}{100} = 80 \text{ min for everyone}$$

Selfish routing plus a new road made every trip 15 minutes longer. Capacity alone does not fix congestion.

Traffic models at three scales

Microscopic

spacing s_n



each vehicle is an ODE
car following

Macroscopic

density $\rho(x, t)$



traffic is a fluid
a PDE, then cells

Network



links, queues, signals
routing and demand

- **Micro** models explain why waves form and how automated vehicles can damp them
- **Macro** models are small enough for real time estimation and control of freeways
- **Network** models describe urban streets, signals, and route choice
- This module spends most of its time on **macroscopic freeway models**

PART 2

Traffic flow fundamentals

Flow, density, and speed, the fundamental diagram, and a short look at car following

Flow, density, and speed

Flow counts vehicles passing a point

$$Q = \frac{\Delta N}{\Delta t} \quad (\text{veh/h})$$

Density counts vehicles on a stretch of road

$$\rho = \frac{\text{vehicles}}{\text{length}} \quad (\text{veh/mi})$$

Speed links the two

$$Q = \rho v$$

At a fixed point, the right average of individual speeds v_n is the **harmonic** mean

$$v_H = \frac{\Delta N}{\sum_n 1/v_n}$$

Example. A detector sees 5 vehicles in 10 s at 60, 60, 30, 30, and 60 mph.

- Flow is $5/10 \times 3600 = 1800$ veh/h
- Arithmetic mean speed is 48 mph
- Harmonic mean speed is $5/(3/60 + 2/30) = 42.9$ mph
- So density is $1800/42.9 = 42$ veh/mi

Using the arithmetic mean gives 37.5 veh/mi and hides 11% of the vehicles. Slow vehicles stay longer on the road, so they deserve more weight.

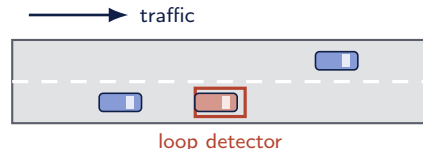
From detectors to density

Density is hard to measure directly, because it needs a snapshot of a whole road section.

- **Loop detectors** report counts and *occupancy*, the fraction of time a vehicle covers the loop
- **Probe vehicles** and phones report speeds along their path
- **Cameras** such as I-24 MOTION report full trajectories, but on short corridors

With an effective vehicle plus loop length L

$$\rho \approx \frac{\text{occupancy}}{L}$$



Example. Occupancy is 10% and $L = 22$ ft, so

$$\rho \approx \frac{0.10 \times 5280}{22} = 24 \text{ veh/mi}$$

Long trucks change L , which is one reason density must be *estimated* rather than read off a sensor.

The fundamental diagram, Greenshields

Greenshields (1935) measured that speed falls linearly with density

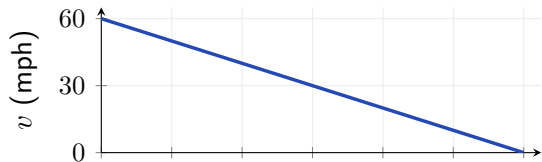
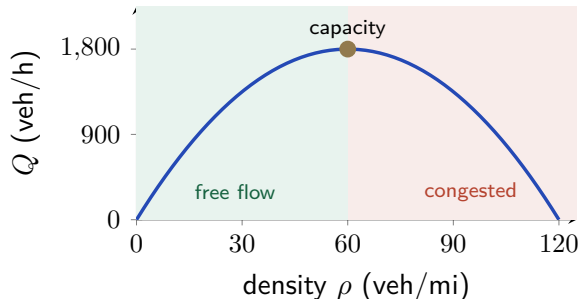
$$v = v_f \left(1 - \frac{\rho}{\rho_m}\right)$$

so flow is a parabola

$$Q_e(\rho) = v_f \rho \left(1 - \frac{\rho}{\rho_m}\right)$$

- Critical density $\rho_c = \rho_m/2$
- Capacity $Q_{\max} = v_f \rho_m/4$

Example. $v_f = 60$ mph and $\rho_m = 120$ veh/mi give $\rho_c = 60$ veh/mi and $Q_{\max} = 1800$ veh/h.



Every flow below capacity can happen at two densities, one fast and one jammed.

The triangular fundamental diagram

Most modern models use two straight lines

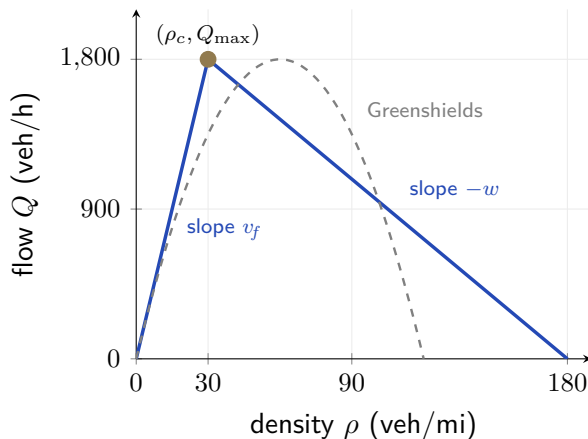
$$Q_e(\rho) = \min(v_f \rho, w(\rho_m - \rho))$$

- Free flow branch with slope v_f
- Congested branch with slope $-w$
- Critical density $\rho_c = \frac{w \rho_m}{v_f + w}$

Example used for the rest of the module, with $v_f = 60$ mph, $w = 12$ mph, and $\rho_m = 180$ veh/mi

$$\rho_c = \frac{12 \times 180}{72} = 30, \quad Q_{\max} = 60 \times 30 = 1800$$

The value $w = 12$ mph matches the backward waves seen on I-24.



A microscopic view, car following in state space

Follower n reacts to its spacing s_n and to the leader's speed v_{n-1} . Linearized around a steady state,

$$\dot{s}_n = v_{n-1} - v_n$$

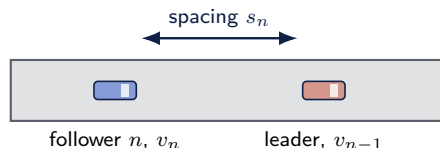
$$\dot{v}_n = \kappa(V' s_n - v_n) + \lambda(v_{n-1} - v_n)$$

κ is a relaxation rate, V' the slope of the desired speed versus spacing, and λ the gain on relative speed.

With state $[s_n \ v_n]^T$ and input $u = v_{n-1}$

$$\begin{bmatrix} \dot{s}_n \\ \dot{v}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ \kappa V' & -(\kappa + \lambda) \end{bmatrix}}_A \begin{bmatrix} s_n \\ v_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ \lambda \end{bmatrix}}_B u$$

Stability of one car is not enough. The next slide asks what happens down a line of cars.



Example. $\kappa = 0.6$, $V' = 0.5$, $\lambda = 0.5$ (per second) give

$$A = \begin{bmatrix} 0 & -1 \\ 0.3 & -1.1 \end{bmatrix}$$

with eigenvalues -0.5 and -0.6 . One follower behind one leader is stable (Module 4).

String stability and phantom jams

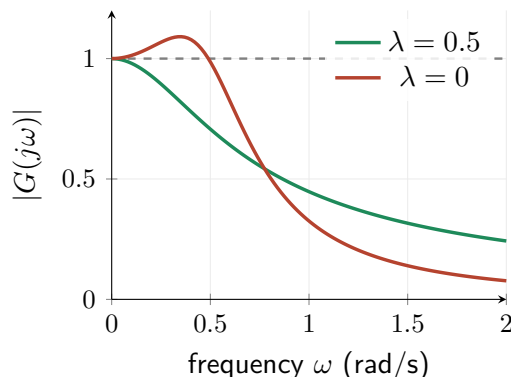
The transfer function from the leader's speed to the follower's speed is

$$G(j\omega) = \frac{\lambda j\omega + \kappa V'}{(j\omega)^2 + (\kappa + \lambda)j\omega + \kappa V'}$$

A platoon is **string stable** when $|G(j\omega)| \leq 1$ at every frequency, so a speed wiggle shrinks down the line. Working out $|\text{den}|^2 - |\text{num}|^2 \geq 0$ gives

$$\kappa + 2\lambda \geq 2V'$$

- With $\lambda = 0.5$ the test gives $1.6 \geq 1$, so it is string stable
- With $\lambda = 0$ it gives $0.6 < 1$, so the peak gain is 1.09, and after 10 vehicles a wiggle is $1.09^{10} = 2.4$ times larger



Drivers who ignore relative speed amplify waves, and that is how a phantom jam is born.

PART 3

LWR and cell transmission

A conservation law for vehicles, shock waves and queues by hand, and a simulator you can write in fifteen lines

Conservation of vehicles

Count vehicles on a short stretch
 $[x, x + \Delta x]$. What enters minus what leaves
 is what accumulates

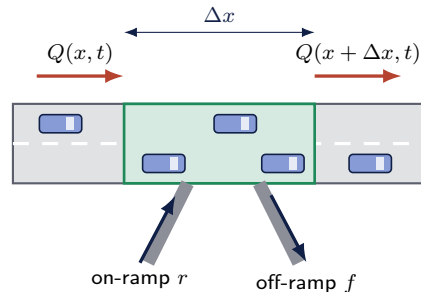
$$\frac{\partial}{\partial t}(\rho \Delta x) = Q(x, t) - Q(x + \Delta x, t) + r - f$$

with on-ramp inflow r and off-ramp outflow
 f . Divide by Δx and let it shrink

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0}$$

on a stretch without ramps.

- This is the same mass balance as a water pipe or a tank in Module 5
- One equation, two unknowns ρ and Q , so a third relation is needed



The LWR model and its wave speed

Lighthill, Whitham, and Richards (1955 and 1956) closed the conservation law with the fundamental diagram $Q = Q_e(\rho)$

$$\frac{\partial \rho}{\partial t} + Q'_e(\rho) \frac{\partial \rho}{\partial x} = 0$$

This is an advection equation, like chlorine in a pipe, but the speed $Q'_e(\rho)$ now depends on the density itself.

Wave speed for Greenshields

$$Q'_e(\rho) = v_f \left(1 - \frac{2\rho}{\rho_m} \right)$$

Free flow sends information downstream, congestion sends it upstream.

Example with $v_f = 60$ and $\rho_m = 120$

- At $\rho = 30$ cars move at 45 mph and the wave at +30 mph, slower than the cars
- At $\rho = 90$ it moves at -30 mph, **backward**

For the triangular diagram the wave speed is $+v_f$ in free flow and $-w$ in congestion, the 12 mph backward wave of the space-time diagram.

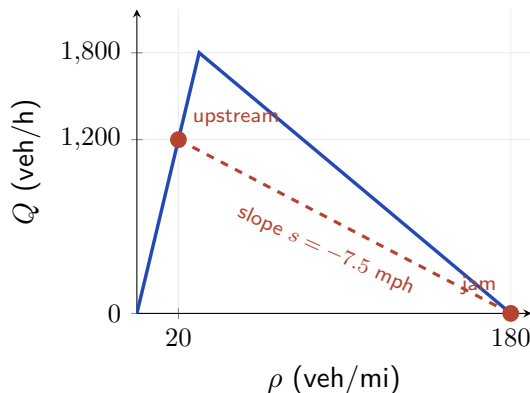
Shock waves

When fast, light traffic runs into slow, dense traffic, density jumps across a moving front, a **shock**. Conserving vehicles across the front gives its speed

$$s = \frac{Q_2 - Q_1}{\rho_2 - \rho_1}$$

the slope of the chord between the two states on the fundamental diagram.

- Upstream state (ρ_1, Q_1) , downstream state (ρ_2, Q_2)
- A negative s means the queue tail grows upstream
- When dense traffic is released into light traffic the front spreads instead, a *rarefaction fan*



Example of an incident queue

An incident blocks the road at $x = 0$ for 15 minutes. Upstream traffic flows at 1200 veh/h, so $\rho_1 = 20$.

Queue tail. Free flow meets the jam ($\rho_m = 180$, $Q = 0$)

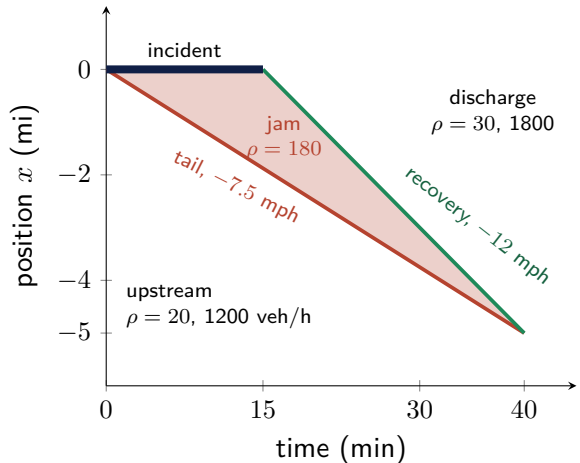
$$s_1 = \frac{0 - 1200}{180 - 20} = -7.5 \text{ mph}$$

Recovery front. After clearing, the queue discharges at capacity (30, 1800)

$$s_2 = \frac{1800 - 0}{30 - 180} = -12 \text{ mph}$$

The two fronts meet when
 $7.5t = 12(t - 0.25)$, so $t = 40$ min at
 $x = -5$ mi.

A 15 minute blockage leaves a 5 mile queue that takes 40 minutes to disappear.



The cell transmission model

Daganzo (1994) split the road into cells of length l and stepped time by T . Each cell keeps a vehicle balance

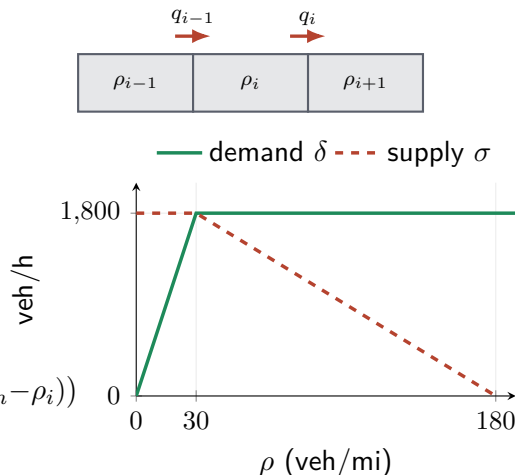
$$\rho_i(k+1) = \rho_i(k) + \frac{T}{l}(q_{i-1}(k) - q_i(k))$$

The flow between cells is the smaller of what the upstream cell can send and the downstream cell can take

$$\delta_i = \min(v_f \rho_i, Q_{\max}), \quad \sigma_i = \min(Q_{\max}, w(\rho_m - \rho_i))$$

$$q_i = \min(\delta_i, \sigma_{i+1})$$

Stability needs $v_f T \leq l$, a vehicle may not jump over a cell in one step (the CFL condition of Module 5).



Example of the cell transmission model by hand

Three cells with $l = 0.5$ mi and $T = 30$ s, so $v_f T = l$ and $T/l = 1/60$ h/mi. Upstream demand is 1500 veh/h and the exit can take only 600 veh/h.

cell	1	2	3
$\rho_i(0)$	40	60	150
demand δ_i	1800	1800	1800
supply σ_i	1680	1440	360

Flows

$$q_0 = \min(1500, 1680) = 1500$$

$$q_1 = \min(1800, 1440) = 1440$$

$$q_2 = \min(1800, 360) = 360$$

$$q_3 = \min(1800, 600) = 600$$

Update

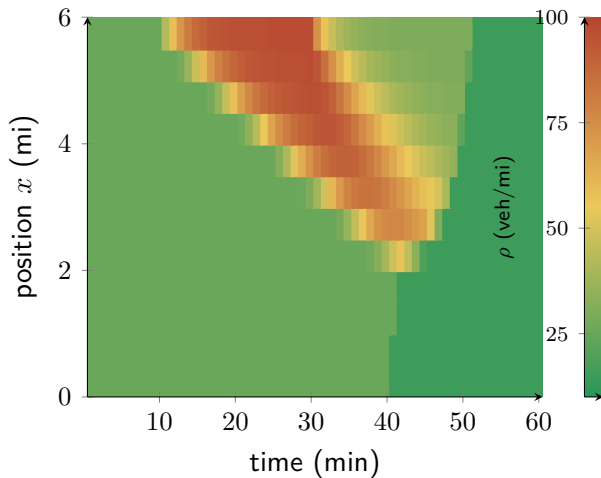
$$\rho_1(1) = 40 + \frac{1500 - 1440}{60} = 41$$

$$\rho_2(1) = 60 + \frac{1440 - 360}{60} = 78$$

$$\rho_3(1) = 150 + \frac{360 - 600}{60} = 146$$

Cell 3 cannot take what cell 2 sends, so cell 2 fills up. The queue grows backward one cell at a time.

A cell transmission simulation of an incident



A 6 mile freeway in 12 cells.

- Demand is 1500 veh/h, falling to 900 after 40 min
- From minute 10 to 30 an incident at the exit lets only 1000 veh/h out
- The queue holds
 $\rho = 180 - 1000/12 = 96.7$ veh/mi, on the congested branch
- Its tail moves back at
 $(1000 - 1500)/(96.7 - 25) = -7$ mph

The cell transmission model in Python

This is the code behind the previous slide.

```
import numpy as np
vf, w, rm, Qmax = 60.0, 12.0, 180.0, 1800.0      # mph, mph, veh/mi, veh/h
l, T, n = 0.5, 30/3600, 12                      # cell (mi), step (h), cells
demand = lambda r: np.minimum(vf*r, Qmax)       # what a cell can send
supply = lambda r: np.minimum(Qmax, w*(rm - r))  # what a cell can take
rho = np.full(n, 25.0)                         # initial densities
for k in range(120):                           # one hour
    q_in  = 1500 if k < 80 else 900              # upstream demand
    q_out = 1000 if 20 <= k < 60 else Qmax       # incident at the exit
    q = np.minimum(np.r_[q_in, demand(rho)], np.r_[supply(rho), q_out])
    rho = rho + T/l*(q[:-1] - q[1:])             # vehicle balance per cell
```

- One line of physics per cell, the rest is bookkeeping
- The `np.minimum` of demand and supply is the only nonlinearity

The cell transmission model in state space

Stack the cell densities into $\rho(k) \in \mathbb{R}^n$. Then

$$\rho(k+1) = \rho(k) + \frac{T}{l} M q(\rho(k), u(k))$$

with M the incidence matrix of the road (Modules 4 and 5) and u the boundary demand and supply.

In free flow every $q_i = v_f \rho_i$, so with $\alpha = v_f T/l$

$$\rho_i(k+1) = (1 - \alpha) \rho_i(k) + \alpha \rho_{i-1}(k)$$

$$A = \begin{bmatrix} 1 - \alpha & & & \\ \alpha & 1 - \alpha & & \\ & \alpha & 1 - \alpha & \\ & & \alpha & 1 - \alpha \end{bmatrix}$$

exactly the upwind pipe of Module 5 with $\kappa = 0$.

In congestion every $q_i = w(\rho_m - \rho_{i+1})$, so information flows the other way and A is *upper* triangular with weight wT/l .

With Greenshields the flux is quadratic, so even the free flow model is nonlinear. The next slides derive it cell by cell and stack it into

$$\dot{x} = Ax + f(x) + B_u u$$

The model switches

Each cell is free or congested, so the CTM is a *switched* linear system. Estimators and controllers must track which mode is active.

Greenshields demand and supply

Godunov's scheme splits the road into cells but keeps time continuous

$$\dot{\rho}_i = \frac{\partial \rho}{\partial t} = -\frac{\partial Q}{\partial x} \approx \frac{q_{i-1} - q_i}{l}$$

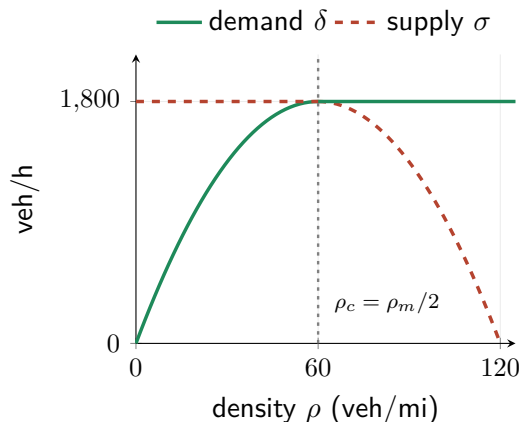
With the Greenshields flux $Q(\rho) = v_f \rho - \frac{v_f}{\rho_m} \rho^2$, a cell's **demand** (what it wants to send) and **supply** (what it can take) are

$$\delta(\rho) = \begin{cases} Q(\rho), & \rho < \rho_c \\ \frac{1}{4}v_f\rho_m, & \rho_c \leq \rho \leq \rho_m \end{cases}$$

$$\sigma(\rho) = \begin{cases} \frac{1}{4}v_f\rho_m, & \rho < \rho_c \\ Q(\rho), & \rho_c \leq \rho \leq \rho_m \end{cases}$$

and the flow between cells is

$$q_i = \min(\delta(\rho_i), \sigma(\rho_{i+1}))$$



$v_f = 60$ mph and $\rho_m = 120$ veh/mi, so
 $\frac{1}{4}v_f\rho_m = 1800$ veh/h.

Removing the min in each traffic mode

The min makes the model switch. It disappears when every cell is in the same mode.

All cells free flowing, $\rho_i < \rho_c$. Every supply is the full $\frac{1}{4}v_f\rho_m$, which is at least any demand, so

$$q_i = \delta(\rho_i) = v_f\rho_i - \frac{v_f}{\rho_m}\rho_i^2$$

All cells congested, $\rho_c \leq \rho_i \leq \rho_m$. Every demand is the full $\frac{1}{4}v_f\rho_m$, so

$$q_i = \sigma(\rho_{i+1}) = v_f\rho_{i+1} - \frac{v_f}{\rho_m}\rho_{i+1}^2$$

In free flow a cell's outflow depends on itself, and in congestion on the cell ahead of it.

Free flow, interior cell. Substitute into

$$\dot{\rho}_i = (q_{i-1} - q_i)/l$$

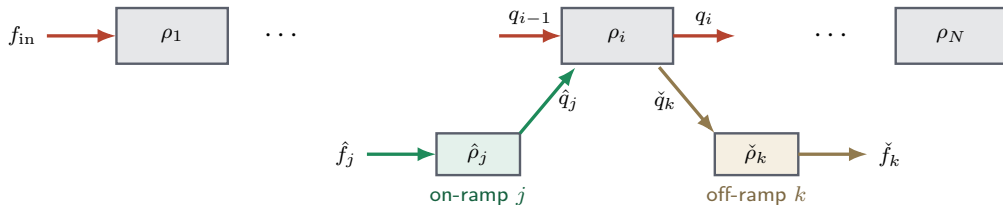
$$\dot{\rho}_i = \frac{v_f}{l}(\rho_{i-1} - \rho_i) - \frac{v_f}{l\rho_m}(\rho_{i-1}^2 - \rho_i^2)$$

With $\beta \triangleq \frac{v_f}{l\rho_m}$

$$\dot{\rho}_i = \underbrace{\frac{v_f}{l}(\rho_{i-1} - \rho_i)}_{\text{linear}} + \underbrace{\beta(\rho_i^2 - \rho_{i-1}^2)}_{\text{quadratic}}$$

The linear part is the upwind transport of Module 5. The quadratic part is the slowdown as density rises.

Greenshields free flow model, cell by cell



Mainline cells, with $Q(\rho) = v_f \rho - \frac{v_f}{\rho_m} \rho^2$

- First cell, $\dot{\rho}_1 = \frac{f_{in}}{l} - \frac{v_f}{l} \rho_1 + \beta \rho_1^2$
- Interior cell,

$$\dot{\rho}_i = \frac{v_f}{l} (\rho_{i-1} - \rho_i) + \beta (\rho_i^2 - \rho_{i-1}^2)$$
- With on-ramp j , add $+\hat{q}_j/l = \frac{v_f}{l} \hat{\rho}_j - \beta \hat{\rho}_j^2$
- With off-ramp k , add

$$-\check{q}_k/l = -\gamma_k \left(\frac{v_f}{l} \check{\rho}_k - \beta \check{\rho}_k^2 \right)$$

Ramp cells

- On-ramp, $\dot{\hat{\rho}}_j = \frac{\hat{f}_j}{l} - \frac{v_f}{l} \hat{\rho}_j + \beta \hat{\rho}_j^2$
- Off-ramp, $\dot{\check{\rho}}_k = \gamma_k \left(\frac{v_f}{l} \check{\rho}_k - \beta \check{\rho}_k^2 \right) - \frac{\check{f}_k}{l}$

$\hat{f}_j$ is the demand entering on-ramp j , $\check{f}_k$ the flow leaving off-ramp k , and γ_k the share coefficient of off-ramp k . A cell with both ramps adds both terms.

Greenshields nonlinear state space model

Stack N mainline cells, N_I on-ramps, and N_O off-ramps into $x = [\rho^\top \quad \hat{\rho}^\top \quad \check{\rho}^\top]^\top \in \mathbb{R}^{N+N_I+N_O}$ and $u = [f_{\text{in}} \quad \hat{f}^\top \quad \check{f}^\top]^\top$. Then

$$\dot{x} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} x + f(x) + B_u u$$

$$A_1 = \frac{v_f}{l} \begin{bmatrix} -1 & & & & \\ 1 & -1 & & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}$$

- A_2 has $+v_f/l$ where on-ramp j feeds cell i , and $-\gamma_k v_f/l$ where cell i feeds off-ramp k
- A_3 is diagonal, $-v_f/l$ for on-ramps and $+\gamma_k v_f/l$ for off-ramps
- B_u has $1/l$ for f_{in} and each $\hat{f}_j$, and $-1/l$ for each $\check{f}_k$

$f(x)$ collects the quadratic terms, row by row

$$f_i(x) = \beta \begin{cases} \rho_1^2 & \text{first cell} \\ \rho_i^2 - \rho_{i-1}^2 & \text{interior cell} \\ \cdots - \hat{\rho}_j^2 & \text{with on-ramp} \\ \cdots + \gamma_k \check{\rho}_k^2 & \text{with off-ramp} \\ \hat{\rho}_j^2 & \text{on-ramp cell} \\ -\gamma_k \check{\rho}_k^2 & \text{off-ramp cell} \end{cases}$$

The zero block says ramps are not affected by mainline density in free flow, so information flows only from ramps into the mainline.

Example of a Greenshields state space model

Three cells with an on-ramp at cell 2, $v_f = 60$ mph, $\rho_m = 120$ veh/mi, $l = 0.5$ mi, so $v_f/l = 120$ and $\beta = 1$ (per hour). State $x = [\rho_1 \ \rho_2 \ \rho_3 \ \hat{\rho}_1]^T$, input $u = [f_{in} \ \hat{f}_1]^T$.

$$A = \begin{bmatrix} -120 & 0 & 0 & 0 \\ 120 & -120 & 0 & 120 \\ 0 & 120 & -120 & 0 \\ 0 & 0 & 0 & -120 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} \rho_1^2 \\ \rho_2^2 - \rho_1^2 - \hat{\rho}_1^2 \\ \rho_3^2 - \rho_2^2 \\ \hat{\rho}_1^2 \end{bmatrix}, \quad B_u = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Equilibrium for $f_{in} = 1200$ and $\hat{f}_1 = 300$ veh/h. Each cell's outflow equals its inflow, so solve $Q(\rho) = q$ on the free flow branch, $\rho = 60 - \sqrt{3600 - 2q}$

state	flow	density
ρ_1	1200	25.36
$\hat{\rho}_1$	300	5.23
ρ_2, ρ_3	1500	35.51

Linearize (Module 4). The Jacobian is $A + \partial f / \partial x$, whose diagonal is $-120 + 2\rho_i$

$$\lambda = -69.3, \ -49.0, \ -49.0, \ -109.5$$

All negative, so free flow is stable and settles in about a minute.

Beyond first order models

LWR assumes speed adjusts instantly to density, so it cannot reproduce the **capacity drop**, where a queue discharges about 10% below capacity, or stop-and-go oscillations.

Second order models add a speed equation. The Aw Rascle Zhang (ARZ) model is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

$$\frac{\partial(\rho(v + p(\rho)))}{\partial t} + \frac{\partial(\rho v(v + p(\rho)))}{\partial x} = \frac{\rho(V_e(\rho) - v)}{\tau}$$

with a traffic pressure $p(\rho) = v_f(\rho/\rho_m)^\gamma$ and relaxation time τ .

- The state per cell doubles, density and relative flow
- The discretized model is again $\rho(k+1) = A\rho(k) + f(\rho, u)$ with twice the states
- A cheaper fix keeps CTM and lowers the demand of a congested cell, $\delta_i = (1 - \epsilon)Q_{\max}$, which the ramp metering example uses

PART 4

Traffic state estimation

Recovering every cell density from a few sensors, and
choosing where those sensors should go

Which cells can a sensor see?

Take three cells in free flow with
 $\alpha = v_f T / l = 0.5$ ($l = 0.5$ mi, $T = 15$ s)

$$\rho(k+1) = \underbrace{\begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}}_A \rho(k) + \underbrace{\begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}}_B u(k)$$

where u is the density arriving from upstream. A single loop detector measures one cell, $y(k) = C\rho(k)$.

Observability (Module 4) with $\mathcal{O}$ stacking C , CA , and CA^2

In free flow a sensor sees only what has passed it, so put it downstream. In congestion the answer flips.



sensor on	rank $\mathcal{O}$
cell 1, $C = [1 \ 0 \ 0]$	1
cell 2, $C = [0 \ 1 \ 0]$	2
cell 3, $C = [0 \ 0 \ 1]$	3

Example of estimating densities by least squares

The detector on cell 3 reports five readings with noise while $u = 20$ veh/mi enters

$$y = (50.6, 39.6, 30.3, 23.25, 20.82)$$

Each reading is linear in the unknown initial densities $\rho(0)$

$$y(k) = CA^k \rho(0) + \underbrace{\sum_{j < k} CA^{k-1-j} Bu}_{\text{known input part}} + \text{noise}$$

Stack the five rows into $Y = \mathcal{O}_5 \rho(0) + G + e$ and solve

$$\hat{\rho}(0) = \arg \min_{\rho(0)} \|Y - G - \mathcal{O}_5 \rho(0)\|^2$$

$$\mathcal{O}_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.375 & 0.375 & 0.125 \\ 0.375 & 0.25 & 0.0625 \end{bmatrix}$$

$$G = (0, 0, 0, 2.5, 6.25)$$

	ρ_1	ρ_2	ρ_3
true	10	30	50
estimate	10.69	29.03	50.57

One detector and five readings recover all three densities, because the system is observable.

Estimators that run in real time

Luenberger observer (Module 4)

$$\hat{\rho}(k+1) = A\hat{\rho}(k) + Bu(k) + L(y(k) - C\hat{\rho}(k))$$

with L chosen so that $A - LC$ is stable.

Kalman filter adds process noise Q_w and sensor noise R_v

$$\hat{\rho}^- = A\hat{\rho} + Bu, \quad P^- = APA^T + Q_w$$

$$L = P^- C^T (CP^- C^T + R_v)^{-1}$$

$$\hat{\rho} = \hat{\rho}^- + L(y - C\hat{\rho}^-)$$

What makes traffic hard

- The model switches between free and congested modes, so the filter must also estimate the mode
- Ensemble and particle filters handle the nonlinear CTM directly
- Moving horizon estimation solves the least squares of the previous slide over a sliding window, with bounds $0 \leq \rho \leq \rho_m$, as a QP (Module 3)
- Physics informed machine learning fits the conservation law to camera and probe data

Where to place sensors

Let $\gamma_j \in \{0, 1\}$ decide whether cell j gets a sensor, so $y = \Gamma C \rho$ with $\Gamma = \text{diag}(\gamma)$. The observability Gramian over N steps is

$$W_o(\gamma) = \sum_{k=0}^{N-1} (A^k)^T C^T \Gamma C A^k$$

Pick sensors to make it large

$$\max_{\gamma} \log \det W_o(\gamma) \quad \text{or} \quad \max_{\gamma} \text{trace } W_o(\gamma)$$

$$\text{subject to} \quad \sum_j \gamma_j \leq p, \quad \gamma \in \{0, 1\}^n$$

Trace is linear in γ , so it ranks locations one by one. Log det rewards sensors that see *different* directions.

Example with the three cell model, one sensor, $N = 10$

sensor on	trace W_o	det W_o
cell 1	1.33	0
cell 2	2.07	0
cell 3	2.61	0.21

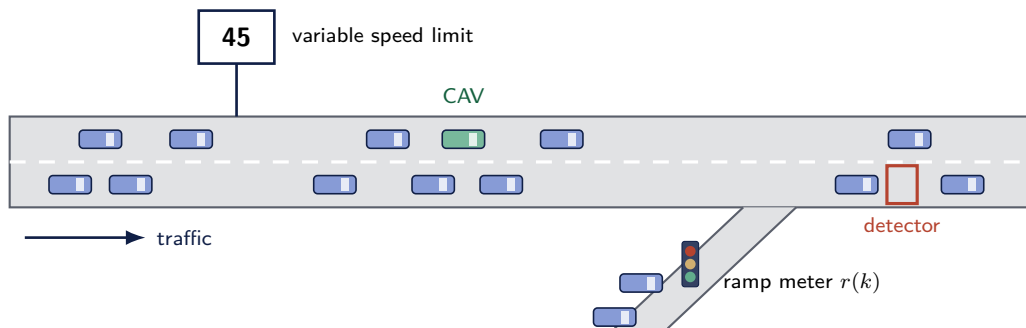
Only cell 3 gives a nonsingular Gramian, matching the rank test. With nonlinear CTM the Gramian is built from Jacobians along a trajectory, and relaxing $\gamma \in [0, 1]$ gives a convex problem.

PART 5

Traffic control

Ramp metering, variable speed limits, automated vehicles,
and traffic signals, each with a worked example

Control levers on a freeway



Lever	Where it enters the CTM	Typical method
Ramp metering	caps the on-ramp inflow, $r(k) \leq r_{\max}$	ALINEA feedback, LP and MPC
Variable speed limit	lowers v_f in chosen cells	shock wave rules, MPC
Connected automated vehicles	act as moving bottlenecks and damp waves	string stable car following
Traffic signals	switch link outflow on and off	queue models, LQ and QP

Why holding cars back can help

A merge has capacity $Q_{\max} = 1800$ veh/h. When a queue forms it discharges only 1620 veh/h, a 10% **capacity drop**.

Mainline demand is 1500 and ramp demand 500 veh/h, so 2000 veh/h arrive at the merge.

No metering

The merge breaks down and serves 1620 veh/h. The queue grows at $2000 - 1620 = 380$ veh/h and spills onto the mainline.

Meter the ramp to 300 veh/h

Total inflow is exactly 1800, so the merge never breaks down and serves 1800 veh/h. Only the ramp queues, at $500 - 300 = 200$ veh/h.

After one hour of peak demand

- 380 versus 200 vehicles are waiting
- The merge serves 11% more vehicles
- The mainline stays free flowing

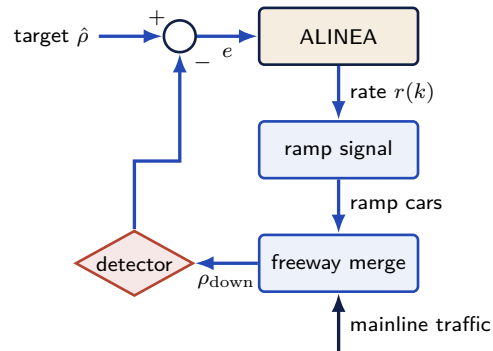
Metering trades a short ramp queue for keeping the bottleneck at full capacity.

What ALINEA is and what it does

ALINEA (French *Asservissement LINéaire d'Entrée Autoroutière*, linear feedback control of a freeway entrance) is the ramp metering law of Papageorgiou, Hadj-Salem, and Blosseville (1991), the most widely deployed local ramp meter.

Every control period (20 to 60 s) it

- ❶ reads the density ρ_{down} from a detector just downstream of the merge
- ❷ compares it with a target $\hat{\rho}$ slightly below ρ_c
- ❸ raises or lowers the metering rate r (veh/h)
- ❹ turns r into ramp signal timing, one car per green, so a car is released every $3600/r$ seconds



One detector, one signal, and one line of arithmetic. ALINEA needs no forecast of demand and no model of the freeway.

ALINEA holds the merge just below critical, so the capacity drop never starts.

The ALINEA control law

The update is an integral controller

$$r(k) = r(k-1) + K(\hat{\rho} - \rho_{\text{down}}(k))$$

- Too dense means meter harder, too light means release more
- The rate is a running sum of past errors, so the merge settles exactly at $\hat{\rho}$ whatever the demand (integral action, Module 4)
- Clip r to $[r_{\min}, r_{\max}]$ and restart from the rate actually applied (anti windup)
- Field versions use loop occupancy,
 $r(k) = r(k-1) + K_R(\hat{o} - o(k))$

Example of one step with $K = 20$ veh/h per veh/mi and $\hat{\rho} = 29$ veh/mi

- The detector reads 31 veh/mi
- r changes by $20 \times (29 - 31) = -40$ veh/h

At steady state $\rho_{\text{down}} = 29$, so the merge carries $60 \times 29 = 1740$ veh/h, close to capacity without breaking down.

With 1400 veh/h on the mainline the ramp gets $r = 1740 - 1400 = 340$ veh/h, one car every $3600/340 = 10.6$ s.

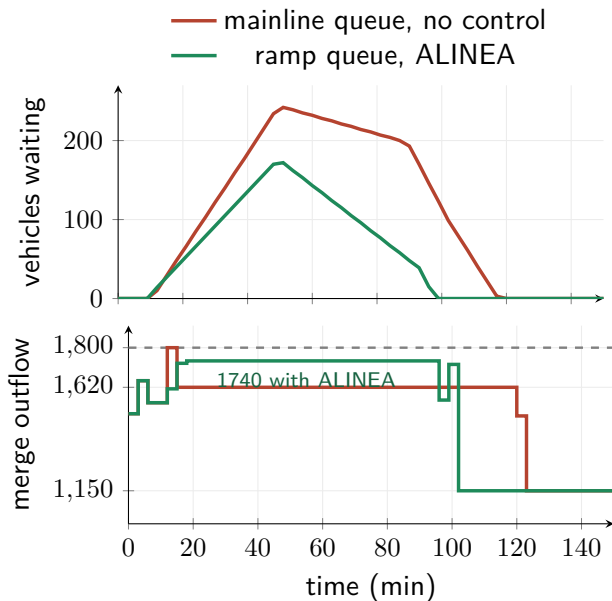
Example of ALINEA on a cell transmission model

Ten cells, an on-ramp at cell 7, capacity drop of 10% at the merge when a queue stands upstream.

- Mainline 1400 veh/h for 90 min, then 1000
- Ramp 600 veh/h from minute 10 to 50, else 150
- ALINEA with $K = 20$ and $\hat{\rho} = 29$

	none	ALINEA
merge outflow	1620	1740
peak vehicles waiting	242	177
congestion ends (min)	120	99
total time spent (veh h)	566	422

Both serve the same 3812 vehicles, but metering cuts total time spent by 25%.



Ramp metering as a linear program

Choose metering rates over a horizon to minimize **total time spent** on the mainline and in ramp queues Q_i^r

$$\min_{\rho, q, r} \sum_k \sum_i T(l \rho_i(k) + Q_i^r(k))$$

subject to, for every cell and step,

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{l} (q_{i-1} + r_i - q_i - f_i)$$

$$Q_i^r(k+1) = Q_i^r(k) + T(d_i^r(k) - r_i(k))$$

$$q_i \leq v_f \rho_i, \quad q_i \leq Q_{\max}, \quad q_i \leq w(\rho_m - \rho_{i+1})$$

$$0 \leq r_i \leq r_{\max}, \quad 0 \leq Q_i^r \leq Q_{\max}^r$$

- Replacing $q_i = \min(\cdot)$ by three inequalities turns the CTM into linear constraints, so the problem is an **LP** (Module 3)
- Under mild conditions the relaxed solution still satisfies the CTM (Gomes and Horowitz, 2006)
- Re-solving every few minutes with new detector data is MPC (Module 4)
- A ramp queue limit $Q_{\max}^r$ keeps cars off local streets

Variable speed limits

A speed limit $v_{\text{lim}} < v_f$ changes the fundamental diagram of a cell

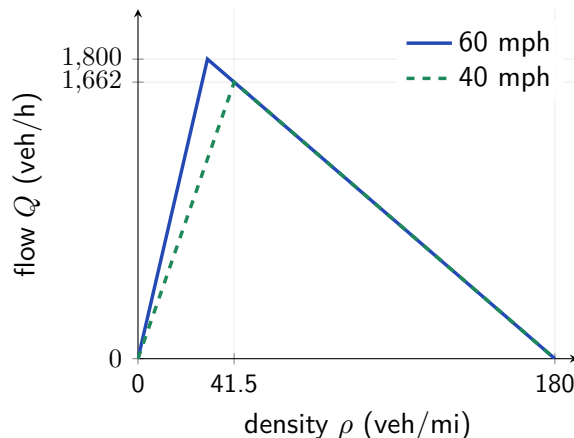
$$Q_e(\rho) = \min(v_{\text{lim}} \rho, w(\rho_m - \rho))$$

Example. Dropping the limit from 60 to 40 mph moves the critical density and capacity

$$\rho_c = \frac{12 \times 180}{40 + 12} = 41.5 \text{ veh/mi}$$

$$Q_{\text{max}} = 40 \times 41.5 = 1662 \text{ veh/h}$$

- A speed limit is a *flow limiter* that acts on the mainline
- Upstream of a jam it slows arrivals so the jam can drain



Example of a speed limit on the incident queue

Return to the 15 minute incident. A 40 mph limit upstream holds arrivals to 900 veh/h, so $\rho_1 = 900/40 = 22.5$.

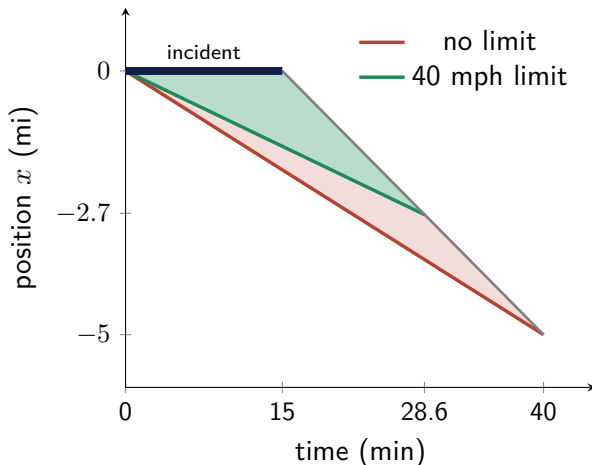
$$s_1 = \frac{0 - 900}{180 - 22.5} = -5.7 \text{ mph}$$

The recovery front still moves at -12 mph and meets the tail when

$$5.7t = 12(t - 0.25) \Rightarrow t = 28.6 \text{ min}$$

	no limit	40 mph
queue length (mi)	5.0	2.7
queue lasts (min)	40	28.6

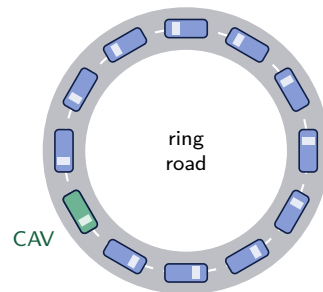
Drivers still wait, but they wait moving at 40 mph instead of stopped in a 5 mile jam.



Automated vehicles as moving actuators

- In the Sugiyama ring experiment (2008), 22 human drivers on a circular track formed a stop-and-go wave with no bottleneck at all
- Stern and colleagues (2018) put one automated vehicle in a similar ring. By driving smoothly it damped the wave and cut fuel use of the whole platoon
- In November 2022 the CIRCLES project ran 100 automated vehicles on I-24 in Nashville to smooth real rush hour traffic

Why it works. The string stability test $\kappa + 2\lambda \geq 2V'$ is a *design rule*. An automated car can choose a large λ that no human driver uses.



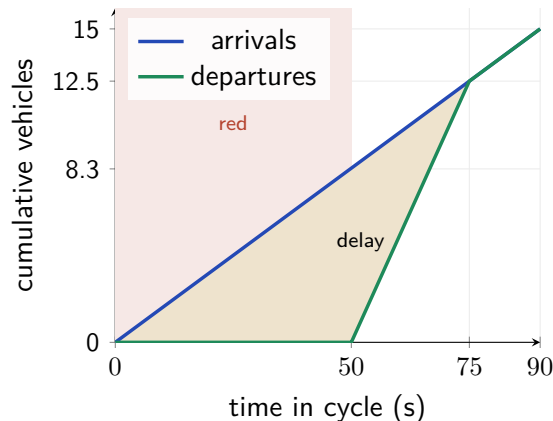
At the macroscopic level a slow automated platoon acts as a *moving bottleneck*, a speed limit that drives with traffic. The model couples the LWR PDE with the vehicle's ODE.

Example of a queue at a signal

Arrivals $a = 600$ veh/h, saturation flow $S = 1800$ veh/h, cycle 90 s with 40 s of green.

- Capacity
 $Sg/C = 1800 \times 40/90 = 800$ veh/h, above demand
- The queue grows during the 50 s red to $600 \times 50/3600 = 8.3$ vehicles
- It clears 25 s into green, when $1800t = 600(50 + t)$
- Total delay is the area between the curves, 312.5 veh s, so each of the 15 vehicles waits 20.8 s on average

This is the vertical queue model, $x(k+1) = x(k) + a(k) - Sg(k)$ while a queue exists.



Signals across a network in state space

The **store and forward** model keeps one queue x_z per link and averages over a cycle C

$$x(k+1) = x(k) + Bg(k) + Td(k)$$

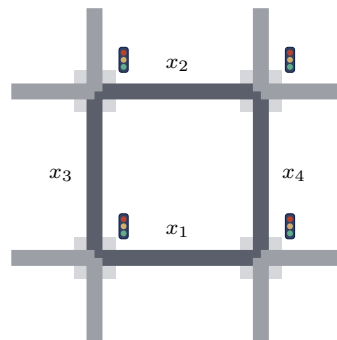
g stacks the green times, and column j of B moves $S_j T/C$ vehicles out of one link and, by turning rates, into the next.

Control as a QP

$$\min_g \sum_k x(k)^T Q x(k) + (g(k) - g_N)^T R (g(k) - g_N)$$

$$\text{s.t.} \quad \sum_{j \in \text{node}} g_j = C - L, \quad g_{\min} \leq g \leq g_{\max}$$

Dropping the constraints gives an LQ regulator, $g(k) = g_N - Lx(k)$, the TUC strategy used in several cities.



S_j is the saturation flow, L the lost time per cycle, and g_N a nominal plan.

Modern directions in transportation systems

Trajectory data

I-24 MOTION near Nashville records every vehicle on four miles of interstate, so models can finally be checked wave by wave.

Automated vehicles

A few well designed cars can damp stop-and-go waves for everyone, as the 100 car CIRCLES test on I-24 showed.

Learning and estimation

Physics informed learning blends the conservation law with camera, loop, and phone data to estimate traffic in real time.

Emissions and energy

Smoothing traffic cuts fuel use, and electric vehicle charging couples traffic with the power grid of Module 4.

Networks and routing

Navigation apps route millions of drivers at once, and Braess's paradox shows why system level routing matters.

Safety

Vision Zero programs use crash data, speed management, and street redesign, all of which are control decisions.

Module recap

Models

- Flow, density, and speed satisfy $Q = \rho v$, and the fundamental diagram $Q_e(\rho)$ closes the model
- Vehicle conservation gives the LWR law $\partial_t \rho + \partial_x Q_e(\rho) = 0$, with waves at $Q'_e(\rho)$
- Shocks move at $(Q_2 - Q_1)/(\rho_2 - \rho_1)$, which predicts queue growth and recovery by hand
- The cell transmission model is a switched state space system, the upwind scheme of Module 5 in free flow
- Car following explains phantom jams through string stability

Estimation

- Downstream sensors see free flow cells, and least squares recovers unmeasured densities
- Gramian scores rank sensor locations

Control

- Ramp metering protects capacity from the capacity drop, by ALINEA feedback or by an LP over the CTM
- Speed limits and automated vehicles shorten queues and damp waves
- Signals form a store and forward state space model controlled by QP or LQ