Your Name: | Your Signature:

- **Exam duration:** 2 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything. You are allowed only one A4-sized sheet of paper.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**.
- If you need more room, use the back of the pages and indicate that you have done so.
- This exam has 20 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules. Good luck! I hope you do well.

Question Number	Maximum Points	Your Score
1	15	
2	15	
3	25	
4	15	
5	10	
6	15	
7	5	
8 (bonus)	5	
Total	100	

1. (15 total points) You are given the following linear optimization problem:

$$\begin{array}{ll} \text{minimize} & 10x_2 + 6x_4 \\ \text{subject to} & x_1 = 7 \\ & 2x_2 + 4x_2 = 2 \\ & x_1 + x_2 + x_4 \geq 0 \\ & x_2 \geq 0 \\ & x_4 \geq 0 \end{array}$$

   (a) (10 points) Write this linear program as a standard LP, i.e., formulate it as

$$\begin{array}{ll} \text{minimize} & c^\top y \\ \text{subject to} & Ay = b \\ & y \geq 0 \end{array}$$

   where y is the new optimization variable, and matrix A and vectors c and b are to be determined (by you). **DO NOT SOLVE this optimization problem**.

(b) (5 points) Write the pseudo code on Matlab's CVX toolbox to solve the above optimization problem.

2. (15 total points) Answer the following unrelated linear algebra questions.

   (a) (5 points) Compute the one-norm $||A||_1$, two-norm $||A||_2$, and infinity norm $||A||_\infty$ of this matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$.

(b) (5 points) For what values of α will the quadratic form

$$f = f(x_1, x_2, x_3) = x^T \begin{bmatrix} 10 & 4 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & \alpha \end{bmatrix} x$$

be positive definite? **Note that this quadratic form is not symmetric.**

(c) (5 points) Compute the eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$ then write the diagonal transformation of matrix $A = TDT^{-1}$ assuming it is has distinct eigenvalues.

3. (25 total points) Answer the following unrelated questions on convexity.

   (a) (5 points) Prove that the set of points whose distance to a does not exceed a fixed fraction β of the distance to b, i.e., the set

   $$\{x \in \mathbb{R}^n, \ \textbf{such that} \ ||x-a||_2 \leq \beta ||x-b||_2\}$$

   for fixed vectors $a, b \in \mathbb{R}^n$ and $\beta \in [0,1]$. To prove this, square both sides of the inequality defining the set and explain why this set is convex for different values of $\beta \in [0,1]$.

(b) (5 points) Is this function $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2$ convex, strictly convex, concave, strictly concave, or none of the above?

(c) (10 points) Compute the linear and quadratic approximations of this nonlinear function

$$f(x) = x_1 x_2 x_3^2 + x_1^2 + 4x_2^2 + 10x_3^3 - x_3^2 x_2^2 + 7x_3$$

around $x^{(0)} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(d) (5 points) Is the quadratic approximation in the previous problem around $x^{(0)}$ a **strictly** convex quadratic approximation?

4. (15 points) The following optimization problem is given:

$$\begin{aligned} \text{minimize} \quad & (x_1 - 2)^2 + 2(x_2 - 1)^2 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 3 \\ & x_1 \geq x_2 \end{aligned}$$

(a) (15 points) Solve this problem via formulating the KKT conditions.

5. (10 points) You are given the following quadratic minimization problem:

$$\underset{x\in\mathbb{R}^2}{\text{minimize}} \quad f(x) = \frac{1}{2}x^\top Qx - x^\top b = \frac{1}{2}x^T \underbrace{\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}}_{Q} x - x^T \underbrace{\begin{bmatrix} -2 \\ 2 \end{bmatrix}}_{b}$$

(a) (10 points) Apply Newton's method (i.e., with step size $t_k = 1$) for the above optimization problem given that the starting point is $x_0 = [0,1]^T$. Perform only two iterations of this method.

6. (15 points) Consider the function

$$f(x_1, x_2) = -2x_1x_2 - 2x_1 + x_1^2 + 2x_2^2.$$

(a) (15 points) Use the method of steepest descent with optimal step-size (exact linear search) to minimize $f(x_1, x_2)$ starting with this initial guess $x^{(0)} = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$. Perform only two iterations of this method.

7. (5 points) You are given the following coupled differential equations:

$$\dot{x}_1(t) = 2x_1(t)x_2(t) + \cos(10t), \quad \dot{x}_2(t) = x_1(t) + x_2(t) + \sin(5t)$$

with initial values $x(t_0 = 0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(a) (5 points) Write a pseudo code for Matlab's ode45 solver to solve these coupled ODEs assuming that the time-span is defined for $t \in [0, 10]$ sec. Your code should reflect both the calling of the ode45 function as well as the Matlab function that encodes the differential equations.

8. (5 points) **[Bonus Problem]**

   (a) (5 points) For what values of the initialization $x^{(0)}$ will Newton's method (with constant stepsize equal to 1) converge for the minimization problem of $f(x) = \sqrt{1+x^2}$? For what values will it diverge?