Your Name:

Your Signature:

- **Exam duration:** 2 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything. You are allowed only one A4-sized sheet of paper.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**.
- If you need more room, use the back of the pages and indicate that you have done so.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules. Good luck! I hope you do well.

Question Number	Maximum Points	Your Score
1	15	
2	15	
3	25	
4	15	
5	10	
6	15	
7	5	
8 (bonus)	5	
Total	100	

1. (15 total points) You are given the following linear optimization problem:

$$\begin{aligned} \text{minimize} \quad & 10x_2 + 6x_4 \\ \text{subject to} \quad & x_1 = 7 \\ & 2x_2 + 4x_3 = 2 \\ & x_1 + x_2 + x_4 \geq 0 \\ & x_2 \geq 0 \\ & x_4 \geq 0 \end{aligned}$$

(a) (10 points) Write this linear program as a standard LP, i.e., formulate it as

$$\begin{aligned} \text{minimize} \quad & c^\top y \\ \text{subject to} \quad & Ay = b \\ & y \geq 0 \end{aligned}$$

where y is the new optimization variable, and matrix A and vectors c and b are to be determined (by you). **DO NOT SOLVE this optimization problem**.

Solution. The constraints can all be rewritten as:

$$\begin{aligned} & x_1 = 7 \\ & 2x_2 + 4x_3 = 2 \\ & x_1 + x_2 + x_4 + s_1 = 0 \\ & x_1 + s_2 = 0 \\ & x_4 + s_3 = 0. \end{aligned}$$

Now, define

$$x_i = x_i^+ - x_i^-, \ i = 1,2,3,4$$

and let $x_i^+ \geq 0, x_i^- \geq 0$, and the slack by definition is non-negative $s_i \geq 0$. Now define variable

$$y^\top = \begin{bmatrix} x_1^+ & x_2^+ & x_3^+ & x_4^+ & x_1^- & x_2^- & x_3^- & x_4^- & s_1 & s_2 & s_3 \end{bmatrix},$$

the optimization problem can now be written as:

$$\begin{aligned} \text{minimize} \quad & c^\top y \\ \text{subject to} \quad & Ay = b \\ & y \geq 0 \end{aligned}$$

where

$$c^\top = \begin{bmatrix} 0 & 10 & 0 & 6 & 0 & -10 & 0 & -6 & 0 & 0 & 0 \end{bmatrix},$$

$$b^\top = \begin{bmatrix} 7 & 2 & 0 & 0 & 0 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(b) (5 points) Write the pseudo code on Matlab's CVX toolbox to solve the above optimization problem.

Solution:

```
A = ... ;
b = ... ;
c = ... ;
cvx_begin
variable y(11)
minimize( c'*y )
subject to
  A * y == b
   y >= zeros(11,1)
cvx_end
```

2. (15 total points) Answer the following unrelated linear algebra questions.

   (a) (5 points) Compute the one-norm $||A||_1$, two-norm $||A||_2$, and infinity norm $||A||_\infty$ of this matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$.

Solution:

- The 1- and infinity-norms are equal to:

$$||A||_1 = 4, ||A||_\infty = 5$$

- As for the 2-norm, you first have to compute matrix $A^\top A = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$. Now, you compute the eigenvalues of this matrix which are $\lambda_1 = \sqrt{116}$ and $\lambda_2 \approx 2.7$. Hence, $||A||_2 = \sqrt{\lambda_{\max}(A)} = \sqrt{\sqrt{116}} \approx 3.65$.

   (b) (5 points) For what values of α will the quadratic form

$$f = f(x_1, x_2, x_3) = x^T \begin{bmatrix} 10 & 4 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & \alpha \end{bmatrix} x$$

be positive definite? **Note that this quadratic form is not symmetric.**

Solution: First, you need to symmetrize the matrix. This results in a symmetric form with $Q = \begin{bmatrix} 10 & 2.5 & 1 \\ 2.5 & 1 & 1 \\ 1 & 1 & \alpha \end{bmatrix}$. This matrix has leading first and second order principal minors that are equal to 10 and 3.75 (both positive). The determinant of this matrix is equal to $10\alpha - 10 - 6.25\alpha + 2.5 + 1.5 > 0$ hence we require $\alpha > \frac{8}{5}$ for the quadratic form to be positive definite.

   (c) (5 points) Compute the eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$ then write the diagonal transformation of matrix $A = TDT^{-1}$ assuming it is has distinct eigenvalues.

Solution: This matrix is the same matrix as in Problem 2-a, so the eigenvalues are $\lambda_1 = \sqrt{116}$ and $\lambda_2 \approx 2.7$. The eigenvectors can be approximated to be $v_1 = \begin{bmatrix} -1.45 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1.45 \end{bmatrix}$ hence the eigendecomposition can be written as

$$A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

3. (25 total points) Answer the following unrelated questions on convexity.

   (a) (5 points) Prove that the set of points whose distance to a does not exceed a fixed fraction β of the distance to b, i.e., the set

$$\{x \in \mathbb{R}^n, \ \textbf{such that} \ ||x-a||_2 \leq \beta ||x-b||_2\}$$

   for fixed vectors $a, b \in \mathbb{R}^n$ and $\beta \in [0,1]$. To prove this, square both sides of the inequality defining the set and explain why this set is convex for different values of $\beta \in [0,1]$.

   Solution: Squaring both sides of the inequality results in

$$(x-a)^\top (x-a) \leq \beta^2 (x-b)^\top (x-b)$$

   which can be expanded into

$$x^\top x + a^\top a - 2x^\top a - \beta^2 x^\top x - 2\beta^2 x^\top b - \beta^2 b^\top b \leq 0.$$

   This last inequality is equivalent to

$$(1-\beta^2) x^\top x - (2a^\top + 2\beta^2 b^\top) x + (a^\top a - \beta^2 b^\top b) \leq 0$$

   which be rewritten as

$$x^\top \left((1-\beta^2) I_n\right) x + h^\top x + c \leq 0$$

   which is an SOCP constraint. For an SOCP constraint to be convex, we require the quadratic weight matrix $Q = (1-\beta^2) I_n$ to be positive semidefinite, which indeed it is for any value of $\beta \in [0,1]$

   (b) (5 points) Is this function $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2$ convex, strictly convex, concave, strictly concave, or none of the above?

   Solution: The Hessian of this function is equal to $\begin{bmatrix} 4 & -10 \\ -10 & 26 \end{bmatrix}$. This matrix has two leading principal minors that are positive. Hence, this function is indeed strictly convex as the Hessian is positive definite.

(c) (10 points) Compute the linear and quadratic approximations of this nonlinear function

$$f(x) = x_1 x_2 x_3^2 + x_1^2 + 4x_2^2 + 10x_3^3 - x_3^2 x_2^2 + 7x_3$$

around $x^{(0)} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Solution: The linear and quaratic approximations are given as follows

$$f_{lin}(x; x^{(0)}) = f(x^{(0)}) + \nabla_x f(x^{(0)})^\top (x - x^{(0)})$$

$$f_{quad}(x; x^{(0)}) = \underbrace{f(x^{(0)}) + \nabla_x f(x^{(0)})^\top (x - x^{(0)})}_{f_{lin}(x; x^{(0)})} + \frac{1}{2}(x - x^{(0)})^\top \nabla_x^2 f(x^{(0)})(x - x^{(0)}).$$

First, compute

$$f(x^{(0)}) = -2, \ \nabla_x f(x^{(0)}) = \begin{bmatrix} 4 \\ 13 \\ 41 \end{bmatrix}, \ \nabla_x^2 f(x^{(0)}) = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 6 & 6 \\ -4 & 6 & -64 \end{bmatrix}$$

Hence

$$f_{lin}(x; x^{(0)}) = -2 + \begin{bmatrix} 4 & 13 & 41 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \\ x_3 + 1 \end{bmatrix}$$

and

$$f_{quad}(x; x^{(0)}) = f_{lin}(x; x^{(0)}) + \frac{1}{2} \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \\ x_3 + 1 \end{bmatrix}^\top \begin{bmatrix} 2 & 1 & -4 \\ 1 & 6 & 6 \\ -4 & 6 & -64 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \\ x_3 + 1 \end{bmatrix}$$

(d) (5 points) Is the quadratic approximation in the previous problem around $x^{(0)}$ a **strictly** convex quadratic approximation?

Solution: The Hessian of the quadratic approximation is given as

$$Q = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 6 & 6 \\ -4 & 6 & -64 \end{bmatrix}.$$

This matrix has a negative determinant (or third order leading principal minor, although positive second and first order leading principal minors), this means this quadratic approximation is *not* strictly convex.

4. (15 points) The following optimization problem is given:

$$\begin{aligned} \text{minimize} \quad & (x_1-2)^2 + 2(x_2-1)^2 \\ \text{subject to} \quad & x_1 + 4x_2 \le 3 \;\equiv\; x_1 + 4x_2 - 3 \le 0 \\ & x_1 \ge x_2 \;\equiv\; x_2 - x_1 \le 0 \end{aligned}$$

(a) (15 points) Solve this problem via formulating the KKT conditions.

Solution: First, we start by formulating the Lagrangian as:

$$\mathcal{L}(x,\mu_1,\mu_2) = (x_1-2)^2 + 2(x_2-1)^2 + \mu_1(x_1+4x_2-3) + \mu_2(x_2-x_1).$$

Then, we can formulate the KKT conditions:

1. $\mu_{1,2} \ge 0$
2. $\mu_1(x_1+4x_2-3) = 0$
3. $\mu_2(x_2-x_1) = 0$
4. $\nabla_x \mathcal{L}(x,\mu_1,\mu_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(x_1-2)+\mu_1-\mu_2 \\ 4(x_2-1)+4\mu_1+\mu_2 \end{bmatrix}$
5. $x_1 + 4x_2 - 3 \le 0$
6. $x_2 - x_1 \le 0$

We now have four cases to consider:

1. Case 1: $\mu_{1,2} = 0$. In that case, we get

$$2(x_1-2) = 0,\ 4(x_2-1) = 0,\ \rightarrow x_1 = 2, x_2 = 1$$

but this solution contradicts the first inequality constraint, this means we have a contradition and this cannot be the solution.

2. Case 2: $\mu_1 = 0, \mu_2 > 0$. You get a contradiction too in this case.
3. Case 3: $\mu_1 > 0, \mu_2 = 0$. In this case, you can write

$$x_1 = 3 - 4x_2$$

then substitute that into $2x_1 - 4 + \mu_1 = 0$. Jointly with $4x_2 - 4 + 4\mu_1 = 0$, these two constraints can be solved for $x_2 = 1/3$ and $\mu_1 = 2/3 > 0$ and hence $x_1 = 5/3$.
This case satisfies all KKT conditions. We now check the Hessian which is

$$\nabla^2 \mathcal{L}(x^*,\mu_1^*,\mu_2^*) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \succ 0$$

meaning that $x^* = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$ is the optimal solution for this problem.

4. Case 4: $\mu_1 = 0, \mu_2 > 0$. You get a contradiction in this case.

5. (10 points) You are given the following quadratic minimization problem:

$$\underset{x\in\mathbb{R}^2}{\text{minimize}} \quad f(x) = \frac{1}{2}x^\top Q x - x^\top b = \frac{1}{2}x^T \underbrace{\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}}_{Q} x - x^T \underbrace{\begin{bmatrix} -2 \\ 2 \end{bmatrix}}_{b}$$

(a) (10 points) Apply Newton's method (i.e., with step size $t_k = 1$) for the above optimization problem given that the starting point is $x_0 = [0,1]^T$. Perform only two iterations of this method.

Solution. The gradient of $f(x)$ is equal to

$$\nabla f(x) = Qx - b = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} x - \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

evaluated at the initial guess, we obtain

$$\nabla f(x^{(0)}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

The Hessian of $f(x)$ is independent on x, which is equal to Q. We can now obtain

$$x^{(1)} = x^{(0)} - t_0 Q^{-1} \nabla f(x^{(0)}) = \begin{bmatrix} -4/3 \\ 10/3 \end{bmatrix}.$$

Evaluating the gradient of $f(x)$ at $x^{(1)}$ yields

$$\nabla f(x^{(1)}) = Qx^{(1)} - b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

meaning that $x^* = \begin{bmatrix} -4/3 \\ 10/3 \end{bmatrix}$ is the solution.

6. (15 points) Consider the function

$$f(x_1, x_2) = -2x_1x_2 - 2x_1 + x_1^2 + 2x_2^2.$$

(a) (15 points) Use the method of steepest descent with optimal step-size (exact linear search) to minimize $f(x_1, x_2)$ starting with this initial guess $x^{(0)} = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$. Perform only two iterations of this method.

Solution: First, we compute the gradient of $f(x)$ as

$$\nabla f(x) = \begin{bmatrix} -2x_2 - 2 + 2x_1 \\ -2x_1 + 4x_2 \end{bmatrix}$$

evaluated at the initial guess, we obtain

$$\nabla f(x^{(0)}) = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

Using the formula for t_0, we obtain

$$t_0 = \arg\min_{t_0>0} f\left(x_0 - t\begin{bmatrix} 6 \\ -6 \end{bmatrix}\right) = \arg\min_{t_0>0} g(t) = \arg\min_{t_0>0}(180t^2 - 72t + 7).$$

Hence, $t_0^* = \dfrac{72}{360} = 0.2$ and note that $g''(t_0^*) = 360 > 0$. We can now obtain

$$x^{(1)} = x^{(0)} - t_0^* \nabla f(x^{(0)}) = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}.$$

Evaluating the gradient of $f(x)$ at $x^{(1)}$ yields

$$\nabla f(x^{(1)}) = \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix}$$

meaning that $x^{(1)}$ is not yet the solution. To compute t_1, we can formulate:

$$t_1 = \arg\min_{t_1>0} f\left(x_1 - t\begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}\right) = \arg\min_{t_1>0} g(t) = \arg\min_{t_1>0}(1.44t^2 - 2.88t - 0.2).$$

The solution is then $t_1^* = 1$ (nice number, eh?). The new iterate then becomes

$$x^{(2)} = x^{(1)} - t_1^* \nabla f(x^{(1)}) = \begin{bmatrix} 1.4 \\ 1 \end{bmatrix}.$$

You can stop here.

7. (5 points) You are given the following coupled differential equations:

$$\dot{x}_1(t) = 2x_1(t)x_2(t) + \cos(10t), \quad \dot{x}_2(t) = x_1(t) + x_2(t) + \sin(5t)$$

with initial values $x(t_0 = 0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   (a) (5 points) Write a pseudo code for Matlab's ode45 solver to solve these coupled ODEs assuming that the time-span is defined for $t \in [0, 10]$ sec. Your code should reflect both the calling of the ode45 function as well as the Matlab function that encodes the differential equations.

Solution: This is one version of the code:

```
function xdot = midterm1_ce4240_problem7(t,x)
xdot = [
 2x(1)* x(2)  + cos(10*t);
  x(1)+ x(2) + sin(5*t) ]
end

tspan = [0 10];
x0 = [-1;1];
[t,X] = ode45(@(t,x) midterm1_ce4240_problem7(t,x), tspan, y0);
```

8. (5 points) **[Bonus Problem]**

   (a) (5 points) For what values of the initialization x_0 will Newton's method (with constant stepsize equal to 1) converge for the minimization problem of $f(x) = \sqrt{1+x^2}$? For what values will it diverge?

Solution: First, you can see that via a simple plot that the minimizer of $f(x)$ is $x = 0$. To see how Newton's method performs, we have to compute the first and second derivatives, as follows:

$$f'(x) = \frac{x}{\sqrt{1+x^2}}, \; f''(x) = \frac{1}{(1+x^2)^{3/2}}.$$

Newton's method iteration can now be written as:

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} = x_k - x_k(1+x_k^2) = -x_k^3.$$

The iterative sequence $x_{k+1} = -x_k^3$ converges to the solution when $-1 < x_0 < 1$ or $|x_0| < 1$. You can see that by trying various values for x_0: the only values that result in convergence are the bounds in the previous bound. And for $|x_0| \geq 1$, Newton's method diverges.