Your Name:

Your Signature:

- **Exam duration:** 2 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything. You are allowed only one A4-sized sheet of paper.
- Solve whatever problems you can solve. The exam is long but (mostly) straightforward. If you've been studying, you'll find it a breather. If you haven't, then I hope the agony is bearable.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**.
- If you need more room, use the back of the pages and indicate that you have done so.
- This exam has 11 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules. Good luck! I hope you do well.

Question Number	Maximum Points	Your Score
1	15	
2	25	
3	15	
4	15	
5	15	
6	15	
Total	100	

1. (15 total points) You are given the following linear optimization problem:

$$
\begin{array}{ll}
\text{minimize} & x_1 + 7x_2 + 4x_3 \\
\text{subject to} & x_1 + 3x_2 + 4x_4 = 7 \\
& 2x_2 + 4x_4 = 2 \\
& x_1 + x_2 + x_3 \geq 0 \\
& x_1 + 2x_2 + 3x_3 \leq 10 \\
& x_1 \geq 0 \\
& x_3 \geq 0
\end{array}
$$

   (a) (15 points) Write this linear program as a standard LP, i.e., formulate it as

$$
\begin{array}{ll}
\text{minimize} & c^\top y \\
\text{subject to} & Ay = b \\
& y \geq 0
\end{array}
$$

   where y is the new optimization variable, and matrix A and vectors c and b are to be determined (by you). **DO NOT SOLVE this optimization problem**.

Solution. The constraints can all be rewritten as follows after adding four slack variables to the four optimization variables (you don't actually need to add slack variables for x_1 and x_3 but let's just do it anyway):

$$\begin{aligned}
&x_1 + 3x_2 + 4x_4 = 7\\
&2x_2 + 4x_4 = 2\\
&-x_1 - x_2 - x_3 + s_1 = 0\\
&x_1 + 2x_2 + 3x_3 + s_2 - 10 = 0\\
&-x_1 + s_3 = 0\\
&-x_3 + s_4 = 0.
\end{aligned}$$

Now, define

$$x_i = x_i^+ - x_i^-,\ i = 1,2,3,4$$

and let $x_i^+ \geq 0, x_i^- \geq 0$, and the slack by definition is non-negative $s_i \geq 0$. Now define variable

$$y^\top = \begin{bmatrix} x_1^+ & x_2^+ & x_3^+ & x_4^+ & x_1^- & x_2^- & x_3^- & x_4^- & s_1 & s_2 & s_3 & s_4 \end{bmatrix},$$

the optimization problem can now be written as:

$$\begin{aligned}
\text{minimize}\quad & c^\top y\\
\text{subject to}\quad & Ay = b\\
& y \geq 0
\end{aligned}$$

where

$$c^\top = \begin{bmatrix} 1 & 7 & 4 & 0 & -1 & -7 & -4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b^\top = \begin{bmatrix} 7 & 2 & 0 & 10 & 0 & 0 \end{bmatrix},$$

and

$$A = \begin{bmatrix}
1 & 3 & 0 & 4 & -1 & -3 & 0 & -4 & 0 & 0 & 0 & 0\\
0 & 2 & 0 & 4 & 0 & -2 & 0 & -4 & 0 & 0 & 0 & 0\\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0\\
1 & 2 & 3 & 0 & -1 & -2 & -3 & 0 & 0 & 1 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

2. (25 total points) Answer the following unrelated questions: *Convexity-and-Linear-Algebra Quick Hits, Taylor's Version.*

   (a) (5 points) Let $\mathcal{S}_+^n$ be the set of symmetric positive semi-definite matrices of size n. Prove that $\mathcal{S}_+^n$ is a convex set.

   Solution. Define two matrices in $\mathcal{S}_+^n$, i.e., $X, Y \in \mathcal{S}_+^n$. You simple need to prove that the convex combination of X and Y, $\alpha X + (1-\alpha)Y$ is also in $\mathcal{S}_+^n$. Well, since $\alpha > 0$ (or between one and zero), then the matrix $Z = \alpha X + (1-\alpha)Y$ will also be a positive definite matrix with non-negative eigenvalues, hence the set $\mathcal{S}_+^n$ is indeed convex.

   (b) (5 points) Is the quadratic form

$$f = f(x_1, x_2, x_3) = x^T \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} x$$

   positive definite, negative semidefinite, or indefinite? Justify your answer carefully. **Note that this quadratic form is not symmetric. Symmetrize it first.**

   Solution. First, we'll have to transform the above quadratic form into a symmetric quadratic one. To do, we can write

$$f = f(x_1, x_2, x_3) = x^T \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} x = \frac{1}{2} x^T \left(\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}^T \right) x$$

   which is equal to

$$f = x^T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x$$

   This matrix has positive and negative and zero principal minors, meaning that this quadratic form is actually indefinite. Also, you can verify that by checking the eigenvalues of A (one is zero, one is negative, and another is positive).

(c) (5 points) Show that the function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} = (x_1^2 + x_2^2)^{\frac{1}{2}}$ is convex. If I were you, I'd use the second order condition.
Hint: In case you forgot, for $g(x) = x^\alpha$, $g'(x) = \alpha x^{\alpha-1}$.

Solution. First, let's compute the gradient and the Hessian of this function. These are computed as:

$$\nabla f(x) = \begin{bmatrix} \frac{1}{2} \cdot 2x_1(x_1^2 + x_2^2)^{-1/2} \\ \frac{1}{2} \cdot 2x_2(x_1^2 + x_2^2)^{-1/2} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{(x_1^2+x_2^2)^{0.5}} \\ \frac{x_2}{(x_1^2+x_2^2)^{0.5}} \end{bmatrix},$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{x_2^2}{(x_1^2+x_2^2)^{1.5}} & \frac{-x_2x_1}{(x_1^2+x_2^2)^{1.5}} \\ \frac{-x_2x_1}{(x_1^2+x_2^2)^{1.5}} & \frac{x_1^2}{(x_1^2+x_2^2)^{1.5}} \end{bmatrix}.$$

To prove that the function is convex, we need to show that the principal minors are all non-negative. The principal minors are indeed all non-negative (the determinant is zero which is fine) and hence the function is indeed convex.

(d) (10 points) Compute the eigenvalues and eigenvectors of $A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution. The eigenvalues of A are $\mathrm{eig}(A) = \lambda_{1,2} = \{-2.23, 2.23\}$. The eigenvectors are:

$$v_1 = \begin{bmatrix} -0.85 \\ 0.52 \end{bmatrix},\ v_2 \begin{bmatrix} 0.52 \\ 0.85 \end{bmatrix}.$$

Approximate solutions are okay!

3. (15 total points) The objective of this problem is to show you how linear matrix inequalities can be written as standard optimization problems. To that end, consider this problem:

$$\begin{array}{lll} \textbf{OP-LMI} & \text{minimize} & \textbf{trace}(P) = p_1 + p_3 \\ & \text{subject to} & -AP - PA^\top \succeq 0 \\ & & P = P^\top \succeq 0 \end{array}$$

where $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ is a constant matrix and $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ is the matrix optimization variable.

   (a) (15 points) Define a new optimization variable $x = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top \in \mathbb{R}^3$ that essentially includes all the variables of matrix P. Given that, write the above optimization problem in standard form as follows:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0, \end{array}$$

   where $f(x)$ is a scalar cost function and $g(x)$ are a multiple constraints to be determined by you. To do so, you need to translate the two constraints in **OP-LMI** into a bunch of (linear/nonlinear) inequality constraints via evaluating positive semi-definiteness of the two constraints. If I'm guessing correctly, you should obtain six inequality constraints (three for each of the two constraints).

Solution. The first constraint can be written as

$$-\left(\begin{bmatrix}-1 & -2\\ 0 & -2\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}+\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}\begin{bmatrix}-1 & -2\\ 0 & -2\end{bmatrix}^\top \succeq 0\right)$$

and the second constraint can be written as

$$\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix} \succeq 0.$$

The first constraint is equivalent to

$$\begin{bmatrix}2p_1+4p_2 & 3p_2+2p_3\\ 3p_2+2p_3 & 4p_3\end{bmatrix} \succeq 0$$

which is equivalent to the following constraints

$$2p_1+4p_2 \geq 0,\ 4p_3 \geq 0,\ (2p_1+4p_2)\cdot(4p_3)-(3p_2+2p_3)^2 \geq 0$$

or

$$2p_1+4p_2 \geq 0,\ 4p_3 \geq 0,\ 8p_1p_3-9p_2^2+4p_2p_3+4p_3^2 \geq 0. \quad (*)$$

These are the first three constraints. The second set of three constraints are a result of $P \preceq 0$ which can be written as three additional constraints:

$$p_1 \geq 0, p_3 \geq 0, p_1p_3-p_2^2 \geq 0. \quad (**)$$

Hence, the optimization problem can be written as

$$\begin{aligned}\text{minimize} \quad & f(x)=c^\top x=\begin{bmatrix}1 & 0 & 1\end{bmatrix}^\top x\\ \text{subject to} \quad & (*),\ (**)\end{aligned}$$

4. (15 points) The following optimization problem is given:

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 + 4x_2^2 \\ \text{subject to} \quad & g_1(x) = 5 - x_1 - x_2 \leq 0 \\ & g_2(x) = 1 - x_2 + x_1^2 \leq 0 \end{aligned}$$

   (a) (15 points) Via formulating and solving the KKT conditions, find *a* solution to the above problem.

Solution. First, we derive the Lagrangian function

$$L(x, \mu) = x_1^2 + 4x_2^2 + \mu_1(5 - x_1 - x_2) + \mu_2(1 - x_2 + x_1^2).$$

Second, we derive the KKT conditions:

- $2x_1 - \mu_1 + 2x_1\mu_2 = 0$
- $8x_2 - \mu_1 - \mu_2 = 0$
- $\mu_1(5 - x_1 - x_2) = 0$
- $\mu_2(1 - x_2 + x_1^2) = 0$
- $\mu_{1,2} \geq 0$

We now consider four different cases:

1. Case 1: $\mu_1 = \mu_2 = 0$:
   This case results in the following constraints:
   - $2x_1 = 0$
   - $8x_2 = 0$
   - $5 - x_1 - x_2 = 0$
   - $1 - x_2 + x_1^2 = 0$

   You get a contradiction here since $x_1 = x_2 = 0$ but that conflicts with the last constraint which is $1 = 0$.
2. Case 2: $\mu_1 > 0, \mu_2 = 0$:
   This case results in the following constraints:
   - $2x_1 - \mu_1 = 0$
   - $8x_2 - \mu_1 = 0$
   - $5 - x_1 - x_2 = 0$
   - $1 - x_2 + x_1^2 < 0$

   Only solution here $x_1 = x_2$ as $\mu_1 = 2x_1 = 8x_2$. This implies that $\mu_1 = 8$, $x_2 = 1$, $x_1 = 4$. So our first candidate solution is $x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, but plugging in this value in the original constraints yields infeasible solution so this case cannot be a solution.

3. Case 3: $\mu_1 = 0, \mu_2 > 0$:
   This case results in the following constraints:
   - $2x_1 + 2x_1\mu_2 = 0$
   - $8x_2 - \mu_2 = 0$
   - $5 - x_1 - x_2 < 0$
   - $1 - x_2 + x_1^2 = 0$

   Solution for this case is $x_1 = 0$ (from the first equation), meaning that $\mu_2 = 8$ and $x_2 = 1$. The second candidate solution is $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but this solution doesn't satisfy the original constraints so this case cannot be a solution.
4. Case 4: $\mu_1, \mu_2 > 0$:
   This case results in the following constraints:
   - $2x_1 - \mu_1 + 2x_1\mu_2 = 0$
   - $8x_2 - \mu_1 - \mu_2 = 0$
   - $5 - x_1 - x_2 = 0$
   - $1 - x_2 + x_1^2 = 0$

   For this system of equations, you get the following two candidate solutions

   $$x^*_{(1)} \approx \begin{bmatrix} -2.6 \\ 7.6 \end{bmatrix}, \; x^*_{(2)} \approx \begin{bmatrix} 1.6 \\ 3.4 \end{bmatrix}.$$

   The first solution yields a negative μ_1, hence it cannot be a solution. The second solution satisfies the constraints. We also need to check the Hessian matrix which is equal to:

   $$\nabla^2_x L(x, \mu) = \begin{bmatrix} -2 + 2\mu_2 & 0 \\ 0 & 8 \end{bmatrix}.$$

   Evaluated at $x^*_{(2)} \approx \begin{bmatrix} 1.6 \\ 3.4 \end{bmatrix}$, we get a positive definite matrix meaning that the point $x^*_{(2)} \approx \begin{bmatrix} 1.6 \\ 3.4 \end{bmatrix}$ is the solution.

5. (15 points) You are given the following quadratic minimization problem:

$$\underset{x\in\mathbb{R}^2}{\text{minimize}} \quad f(x) = \frac{1}{2}x^T \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} x - x^T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   (a) (15 points) Apply *unguarded* Newton's method (i.e., with step size $t_k = 1$) for the above optimization problem given that the starting point is $x_0 = [0, 1]^T$. Only compute a maximum of two iterations.

Solution. The gradient of $f(x)$ is equal to

$$\nabla f(x) = Qx - b = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} x - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

evaluated at the initial guess, we obtain

$$\nabla f(x^{(0)}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

The Hessian of $f(x)$ is independent on x, which is equal to Q. We can now obtain

$$x^{(1)} = x^{(0)} - t_0 Q^{-1} \nabla f(x^{(0)}) = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}.$$

Evaluating the gradient of $f(x)$ at $x^{(1)}$ yields

$$\nabla f(x^{(1)}) = Qx^{(1)} - b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

meaning that $x^{(1)}$ is the solution.

6. (15 points) Consider the function

$$f(x_1, x_2) = 0.5x_1^2 + x_2^2 - x_1 - x_2 + 7.$$

(a) (15 points) Use the method of steepest descent with step-size $t_k = 1$ to minimize $f(x_1, x_2)$ starting with this initial guess $x^{(0)} = \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix}^T$. Only compute a **maximum** of two iterations of this method.

Solution. First, we can write the objective function as

$$f(x) = \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x - \begin{bmatrix} 1 & 1 \end{bmatrix} x + 7 = \frac{1}{2}x^T Q x - x^T b + c$$

The gradient of $f(x)$ is equal to

$$\nabla f(x) = Qx - b$$

evaluated at the initial guess, we obtain

$$\nabla f(x^{(0)}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Using the closed form expression of t_0 (the step-size, or just use the one given in the problem), we obtain

$$t_0 = \frac{\nabla f^T(x^{(0)}) \nabla f(x^{(0)})}{\nabla f^T(x^{(0)}) Q \nabla f(x^{(0)})} = 1$$

We can now obtain

$$x^{(1)} = x^{(0)} - t_0 \nabla f(x^{(0)}) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

Evaluating the gradient of $f(x)$ at $x^{(1)}$ yields

$$\nabla f(x^{(1)}) = Qx^{(1)} - b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

meaning that $x^{(1)}$ is the solution.