HOMEWORK # 3

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Due date of the homework: September 21st, 2023, 11am.

1. Compute the gradient and the Hessian of this function

$$f(x) = f(x_1, x_2, x_3) = 3x_1^2x_3 + 2x_3 - 4x_3x_2^2$$

at this operating point $x^{(0)} = [3\ -1\ 2]^\top$.

2. Compute the gradient and the Hessian of this function

$$f(x) = f(x_1, x_2) = 3x_1^2cos(e^{-x_2}) + 2x_2^2 + 7x_1 - 8x_2 + x_1^2 + 4$$

at this operating point $x^{(0)} = [0\ -1]^\top$.

3. For Problems 1 and 2, do the following:

   (a) Compute the first order and second (quadratic) order Taylor series approximation around the given operating points.

   (b) Using Matlab, plot the functions and their corresponding approximations.

   (c) Evaluate the definiteness of the quadratic approximations (i.e., are the resulting quadratic approximation positive definite, positive semidefinite, negative definite, etc...).

4. Compute the eigenvalues, eigenvectors, determinant, and rank of these two matrices

$$A = \begin{bmatrix} 1 & \pi \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The computations should be by hand but please verify your answers via Matlab.

5. Verify that the one-norm given by

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

is indeed a norm on $\mathbb{R}^n$ via proving that it satisfies the three vector norm properties.

6. Prove that for all vectors $x \in \mathbb{R}^n$, we have

$$||x||_1 \geq ||x||_2.$$

7. Compute $||A||_F$, $||A||_2$, $||A||_1$ and $||A||_\infty$ of this matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}.$$

8. Prove that this new norm for any square matrix A with dimension n defined as

$$||A||_{**} = \max|a_{ij}|, \quad \text{for all } 1 \leq i, j \leq n$$

is not a legit norm. You can find a counter example that contradicts one of the basic matrix norm properties.

9. Prove that the Frobenius norm of any matrix A is indeed a legitimate matrix norm by showing that it satisfies the basic matrix norm properties.

10. Find the *full* and *thin* singular value decompositions for these two matrices by hand and verify them via Matlab

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

Slide 40/56 in Module 2 explains how to compute these decompositions.

11. Compute the 2- and nuclear-norms of matrices A and B given in the previous problems (by utilizing the results of your SVD).

12. In this exercise, I want you to learn how to use Matlab's `ode45` solver. To do so, please visit this link https://www.dropbox.com/scl/fi/39qm4z9wu5li9ygzc72ms/Simulation-with-Matlab.pdf?rlkey=91uii12rxchxg4r0vhdkrtmu5&dl=0 and go through Slides 18–35.

    (a) Once you're done reading/experimenting with Matlab, I want you to produce the plots for the exercise on Slide 33 (Case Study Simulation Model). Essentially, I want you to show me plots for all of the system states.

    (b) Please make sure that your plots are included in the homework solutions and that the plots are clear, not screenshots of figures from Matlab. This link includes details on how to create nice plots (https://shorturl.at/mCOSX). It is a bit advanced though.