HOMEWORK # 4

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Due date of the homework: October 6, 2024, midnight.

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1. Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

   Make a sketch of the feasible set (Matlab can do that for you). For each of the following objective functions, give the optimal set and the optimal value using CVX.

   (a) $f_0(x_1, x_2) = x_1 + x_2$.
   (b) $f_0(x_1, x_2) = -x_1 - x_2$.
   (c) $f_0(x_1, x_2) = x_1$.
   (d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.
   (e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

2. Using the KKT conditions discussed in class, obtain all the candidate strict local minima for the following nonlinear optimization problem:

$$\begin{array}{ll} \text{maximize} & -x_1^2 - 2x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & x_2 - x_1^2 \geq 1 \end{array}$$

   There are many cases to consider. Make sure that you don't miss any.

3. Using the KKT conditions discussed in class, obtain all the candidate strict local minima for the following nonlinear optimization problem:

$$\begin{array}{ll} \text{minimize} & x_1 + x_2^2 \\ \text{subject to} & x_1 - x_2 = 5 \\ & x_1^2 + 9x_2^2 \leq 36 \end{array}$$

   There are many cases to consider. Make sure that you don't miss any.

4. KKT this problem:

$$\begin{array}{ll} \text{minimize} & -x_1 x_2 \\ \text{subject to} & x_1 + x_2^2 \leq 2 \\ & x_{1,2} \geq 0 \end{array}$$

   but also draw the feasible space and try find the optimal solution graphically. Then confirm via CVX, assuming this is a convex problem. Is it?

5. Consider the function

$$f(x_1, x_2) = 0.5x_1^2 + x_2^2 - x_1 - x_2 + 7$$

   Use the Newton's method to minimize $f(x_1, x_2)$ with fixed time step size $t = 1$ and with initial guess $x^{(0)} = [0 \ \ \frac{1}{2}]^\top$.

6. Consider an unknown function whose gradient is given as follows

$$\nabla f(x_1, x_2) = \begin{bmatrix} 8x_1 - 4x_2 \\ -4x_1 + 4x_2 \end{bmatrix}$$

Compute the first two iterations of the steepest descent algorithm for the above function with a given gradient. You can consider that the starting point is $x^{(0)} = [2 \;\; 3]^\top$. Since $f(.)$ is unknown you are not allowed to use any closed form expression for $f(.)$. Use only its gradients. Test this algorithm with various step sizes t.

Once you are done, test different values for t (i.e., $t = 1, 0.95, 0.9, \ldots, 0.1$) via automating this procedure via a for loop. What is the optimal value of t that results in the best convergence towards a stationary point or a local minimum? A figure showcasing this would be nice.

7. Consider the function

$$f(x_1, x_2, x_3) = (x_1 - 4)^4 + (x_2 - 3)^2 + 4(x_3 + 5)^4$$

Use the steepest descent method to minimize $f(x_1, x_2)$ with exact line search and with initial point $x^{(0)} = [4 \;\; 2 \;\; -1]^\top$. Perform three iteration and leave a comment with what you notice.

Once you are done, we want you to write a simple for loop that simulates steepest descent. How many iterations does it take to reach a solution with an error tolerance of 10^{-6}? The error tolerance is defined herein as

$$||x^{(k+1)} - x^{(k)}|| \leq \text{tol} = 10^{-6}$$

where $|| \cdot ||$ is the norm of the error between two successive iterations of the steepest descent method.