Due date: December 1st, @ 11:59pm. This homework contains some mathematical problems that you need to type and programming problems. Upload a single PDF file that shows your codes as well as the solution to the mathematical derivations.

Problem 1 — Hydraulic Simulation of a Simplified Water Distribution Network

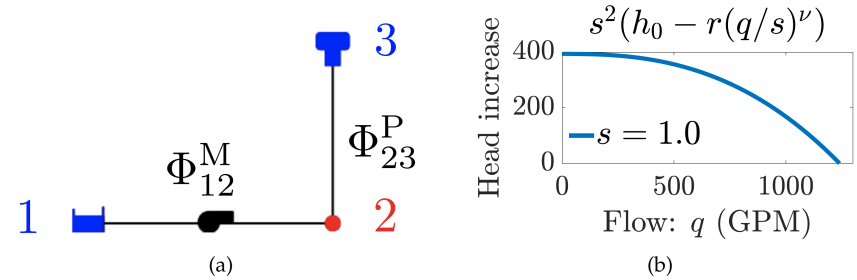

Figure 1: A simplified water distribution network: (a) toplogy of the three-node network, and (b) the corresponding pump head increase curve of Pump 12.

Consider a three-node network containing a reservoir, a junction, a tank, a pump, and a pipe in Fig. 1a. The heads at Reservoir 1, Junction 2, and Tank 3 are denoted by h_1^{R}, h_2^{J}, and h_3^{TK}; the flow rates in Pump 12 and Pipe 23 are denoted by q_{12} and q_{23}. The pump head increase curve

$$\Phi_{12}^{\mathrm{M}}(q_{12}, s) = \Delta h_{12}^{\mathrm{M}}(q_{12}, s) = s^2(h_0 - r(q_{12}/s)^\nu)$$

is given in Fig. 1b. The head loss of Pipe 23 is described by Hazen-Williams equation:

$$\Phi_{23}^{\mathrm{P}}(q_{23}) = Rq_{23}|q_{23}|^{\mu-1}.$$

The dynamics in Tank 3 are given by

$$h_3^{\mathrm{TK}}(k+1) = h_3^{\mathrm{TK}}(k) + \alpha q_{23}.$$

The detailed parameters are given as below.

- Reservoir 1 has fixed head (pressure), that is $h_1^{\mathrm{R}}(k) = 700$ ft, and time-steps $k = 1, 2, \ldots$.
- Pump 12 is on and speed $s = 1$. The parameters in pump increase curve are $h_0 = 393.7$ ft, $r = 3.746 \times 10^{-6}$, and $\nu = 2.59$.
- Junction 2 consumes 240 GPM. That is, $d_2(k) = 240$ GPM, and $k = 1, 2, \ldots$.
- Resistance coefficient $R = 0.035$ in Pipe 23, and flow exponent $\mu = 1.852$
- The initial head in Tank 3 is 908 ft, that is $h_3^{\mathrm{TK}}(0) = 908$ ft, and the head change rate $\alpha = 0.4479$ ft $\cdot$ hour per gallon.
- Each time-step is equivalent to one minute in water hydraulic simulation.

Please solve the following problems and include the MATLAB codes when necessary:

1. List hydraulic models for all five components (mass and energy balance equations).

2. Given parameters above, what are the flow rates in Pump 12 and Pipe 23 when $k = 0$?

3. What is the head at Tank 3 when $k = 1$? Please show your work instead of giving a simple answer.

4. Express hydraulic models as a difference algebraic equation (DAE). That is,

$$\boldsymbol{E}\boldsymbol{x}(k+1) = \boldsymbol{A}_x \boldsymbol{x}(k) + \boldsymbol{E}_d \boldsymbol{d}(k) + \boldsymbol{E}_\Phi \boldsymbol{\Phi}(\boldsymbol{x}(k)),$$

   where $\boldsymbol{x}(k)$ is defined by

$$\boldsymbol{x} = [h_1^{\mathrm{R}}(k), h_2^{\mathrm{J}}(k), h_3^{\mathrm{TK}}(k), q_{12}(k), q_{23}(k)]^\top.$$

   The nonlinear vector function $\boldsymbol{\Phi}(\boldsymbol{x}(k))$ should include all nonlinear head loss/gain.

5. What is the initial condition of the DAE you formulated in Question 4?

6. Simulate the hydraulic dynamics (i.e., DAE) using Matlab. You can write your own solver for difference algebraic equations; or you can convert your difference algebraic equations into differential algebraic equations (**Hint:** the difference $h_3^{\mathrm{TK}}(k+1) - h_3^{\mathrm{TK}}(k)$ can be viewed as a differential term $\dot{h}_3^{\mathrm{TK}}$). Then, there are multiple solvers that are available in Matlab; see `https://www.mathworks.com/help/matlab/math/solve-differential-algebraic-equations-daes.html` for details. Please include your Matlab codes for this question.
   Hints:

   - To solve as difference algebraic equations, use fsolve and formulate the nonlinear function as follows

$$F = -\boldsymbol{E}\boldsymbol{x}(k+1) + \boldsymbol{A}_x \boldsymbol{x}(k) + \boldsymbol{E}_d \boldsymbol{d}(k) + \boldsymbol{E}_\Phi \boldsymbol{\Phi}(\boldsymbol{x}(k)).$$

     Hint inside the hint: define the variable vector to be $\underbrace{\begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{bmatrix}}_{\boldsymbol{x}_{new}(k)}$ and accordingly concatenate the matrices $-\boldsymbol{E},\ \boldsymbol{A}_x$ together.

   - To solve as differential algebraic equations, define a mass matrix with only one entry of 1 corresponding to $h_3^{\mathrm{TK}}(k+1) - h_3^{\mathrm{TK}}(k)$. That is, the nonlinear function should be

$$\tilde{F} = \tilde{\boldsymbol{A}}_x \boldsymbol{x}(k) + \boldsymbol{E}_d \boldsymbol{d}(k) + \boldsymbol{E}_\Phi \boldsymbol{\Phi}(\boldsymbol{x}(k)),$$

     note that $\tilde{\boldsymbol{A}}_x$ is the updated version of $\boldsymbol{A}_x$ to remove the element multiplied by $h_3^{\mathrm{TK}}(k)$.

7. Use the code you built to obtain the water head in Tank 3 after 60 minutes (i.e., 60 time-steps and $k = 60$) and also at 600 minutes. What do you observe?
   Hints:

   - If you built solver for difference algebraic equations, run the solver inside a for loop set to run every minute and updates the solvers parameters. The total time span of the loop should be 600 minutes and do not forget to report the head at 60 minutes as well.

   - If you built solver for differential algebraic equations, define the tspan to be $[0,\ 60,\ 600]$ to report the results at these specific times (you can set it to $[0:1:600]$ and obtain final t if you want to plot the results).

Problem 2 — Water Flow Problem via an Approximate Approach

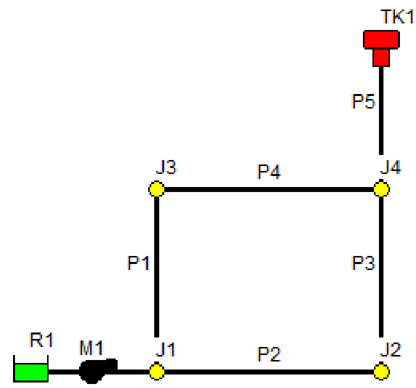

Figure 2: Five-node water distribution network.

Consider the network shown in Fig. 2 that consists of a reservoir, 4 junctions, a tank, a pump, and 5 pipes. The heads at Reservoir R1, Junctions $\{J1, J2, J3, J4\}$, and Tank TK1 are donated by $h^{R1}, h^{J1}, h^{J2}, h^{J3}, h^{J4}, h^{TK1}$; the flow rates in Pump M1 and Pipes $\{P1, P2, P3, P4, P5\}$ are donated by $q^{M1}, q^{P1}, q^{P2}, q^{P3}, q^{P4}, q^{P5}$.

In this problem, you are required to solve a Water Flow Problem using CVX or YALMIP. To be able to do so and to make life easier, the pump head increase curve and pipes head loss equations are approximated to linear formulations as follows,

$$\Delta h^{M1}(q^{M1}) = h_0 - \tilde{r} q^{M1},$$

$$\Delta h^{P}(q^{P}) = \tilde{R} q^{P1}.$$

The rest of the network's components models are unchanged. Consider the detailed parameters to be:

- Reservoir R1 has fixed head (pressure), that is $h^{R1}(k) = 700$ ft.
- The parameters in pump increase curve are $h_0 = 432.8$ ft and $\tilde{r} = 0.324$.
- Demands at junctions are equal to 0, 180, 200, and 150 GPM, respectively to junctions order.
- Resistance coefficient $\tilde{R} = 1.7 \times 10^{-4}$ for all pipes.
- The initial head in Tank TK1 is 915 ft, that is $h^{TK1}(0) = 915$ ft, and the head change rate $\alpha = 0.4479$ ft $\cdot$ hour per gallon.
- Each time-step is equivalent to one minute in water hydraulic simulation.

Solve the following problems and include the MATLAB codes whenever used (pretty figures are considered bonus).

1. Formulate the WFP as an optimization problem and code it MATLAB using CVX for this specific network.
2. What is the head at Tank TK1 when $k = 1$?
3. What is the head at Tank TK1 when $k = 60$? Plot the head from $k = 1$ to $k = 60$.
   Hint: Put the formulated cvx problem in a for loop and update tank head each time-step.
4. Change the demand at Junction J4 to 550 GPM. What is the head at Tank TK1 when $k = 60$? Plot the head from $k = 1$ to $k = 60$.
5. **Bonus:** For the whole simulation window, demand at Junction J4 is set to be 150 GPM, except for $k = 20, 21, 22, 23, 24$ the demand is changed to be $240, 400, 500, 330, 200$ GPM, respectively. What is the corresponding head at Tank TK1 when $k = 60$? Plot the head from $k = 1$ to $k = 60$.

Problem 3 — Quality Simulation of a Simple Water Network

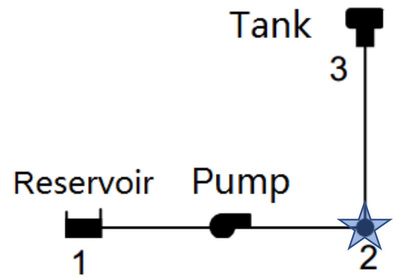

Figure 3: A simplified three-node water distribution network (the blue star is a booster station).

The same simplified three-node network is shown in Fig. 3. Necessary parameters for this quality dynamic simulation are as below.

- Reservoir 1 has fixed concentration, i.e., $c_1^{\mathrm{R}} = 1.0$ mg/L. Initial concentrations in pipes and tanks are 0.2 mg/L.
- Junction 2 consumes water in the rate of $d_2 = 240$ GPM.
- Time-step $\Delta t = \frac{10}{3}$ seconds, and chlorine decay are not considered in this simple network.

Please do the following and include the corresponding MATLAB codes when necessary:

1. If $q_{12} = 923.9$ GPM and $c_{12} = 1.0$ mg/L, then what is the chlorine concentration at Junction 2? That is, $c_2^{\mathrm{J}} =?$ mg/L. Show your work and/or code.
2. If the water is flowing into Tank 3 with fixed flow rate of 683.9 GPM (i.e., $q_{23} = 683.9$ GPM) and fixed chorine concentration of 0.2 mg/L (i.e., $c_{23} = 0.2$ mg/L), and the initial volume in Tank 3 is assumed as $V_3^{\mathrm{TK}} = 2.016 \times 10^4$ ft^3 with concentration of $c_3^{\mathrm{TK}} = 0.1$ mg/L, then what is the concentration in Tank 3 after 20 minutes? List the quality dynamics in Tank 3, and show your Matlab code of simulating the tank dynamics please.
3. To make life easier, Pipe 23 is split into three segments. Let $\alpha = 0.878$ and $r_{23} = 0$ (no decay) for pipes. Please list the quality models of Pipe 23 according to upwind discretization scheme and rewrite the models into matrix form.
4. Find the LDE model for this simple network given the following:
   - Flow rates are fixed. That is, $q_{12} = 923.9$ GPM and $q_{23} = 683.9$ GPM; parameters of tanks are in Question 2
   - Pipe 23 split in 3 segments; parameters in pipes are the same as the ones in Question 3.
   - The flow rate of the booster station in Fig. 3 is 10 GPM, and the corresponding chlorine rate is 5 mg/L.

   Hint:

   - $\chi(t)$ can be defined as $[c_2^{\mathrm{J}}(t), c_1^{\mathrm{R}}(t), c_3^{\mathrm{TK}}(t), c_{23}^{\mathrm{P}}(1,t), c_{23}^{\mathrm{P}}(2,t), c_{23}^{\mathrm{P}}(3,t), c_2^{\mathrm{M}}(t)]^\top$
5. We provide all necessary hydraulic variables (i.e., heads and flows) for you to build the water quality LDE models in the above four questions. In this question, you are encouraged to obtain these parameters by yourself. Please solve the hydraulic dynamics, which is Problem 1, first and use the variables found to compute the LDE models. Simulate the quality dynamics and **find the concentration $\chi(t)$ when $t = 60$ minutes** given the following:
   - The flow rate of the booster station in Fig. 3 is 10 GPM, and the corresponding chlorine rate is 5 mg/L.

- The initial concentrations in pipes and tanks are 0.2 mg/L.
- Reservoir 1 has fixed concentration of 1 mg/L.
- The initial volume in Tank 3 is assumed as $V_3^{\mathrm{TK}} = 2.016 \times 10^4$ ft^3.
- To make the upwind discretization scheme more accurate, you are suggested to split Pipe 23 into 150 segments. Consider $\alpha = 0.878$ and $r_{23}(c^{\mathrm{P}}(t)) = -6.4 * 10^{-3} \times c^{\mathrm{P}}(t)$.

Show your code and plot time series of the chlorine evolution for this time window for Tank 3. Also, plot the evolution for the first, mid, and last segments of the pipe on the same plot and comment.

Hints:

- Hydraulic time-step is taken one minute, while the water quality time-step is $\Delta t = \frac{10}{3}$ seconds.
- Each hydraulic time-step the flows and heads should be updated (i.e., the hydraulic dynamics model is solved) and accordingly the matrices in the water quality LDE models should be updated.
- To facilitate the dependency of the water quality dynamics on the hydraulic dynamics, build a for loop set to run every minute. This loop solves the hydraulic dynamics model. Within this hydraulic loop, establish another nested for loop that updates and solves the water quality LDE models depending on the hydraulic variables. This inner loop for water quality should span the number of water quality time-steps within each hydraulic time-step. Be sure to define the necessary variables to track concentration changes in tanks and pipes throughout the entire simulation to be able to plot them.