VANDERBILT UNIVERSITY
CE 4240/5240
INTRO TO INFRASTRUCTURE SYSTEMS ENGINEERING

HOMEWORK # 8
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This is the final homework of the semester. Been really fun but the fun isn't really over yet. What's going to soon be over is the suffering.

Deadline of this homework is **December 16th at 6pm CT**, as I need to upload the grades on December 17th in the morning my time.

Solve the following two independent problems.

1. Microscopic traffic dynamics can be written as follows:

$$\begin{aligned} \Delta\dot{x}_i &= v_{i-1} - v_i \\ \dot{v}_i &= \alpha_1 v_i + \alpha_2 \Delta x_i + \alpha_3 \Delta\dot{x}_i + \alpha_4 \end{aligned}$$

where $\alpha_1 = -0.253$, $\alpha_2 = 0.23$, $\alpha_3 = 0.07$, $\alpha_4 = -1.61$.

(a) Complete the state-space representation of the linear car-following model discussed in class and given above.You need a few states to represent the system as $\dot{z}(t) = Az(t)$ where $z(t) \in \mathbb{R}^n$ is the state of the system, where n is the number of states that you have to figure out.

$$\dot{z}(t) = Az(t) + B$$

$$z(t) = \begin{bmatrix} v_i \\ \Delta x_i \\ \Delta\dot{x}_i \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha_4 \\ 0 \\ 0 \end{bmatrix}$$

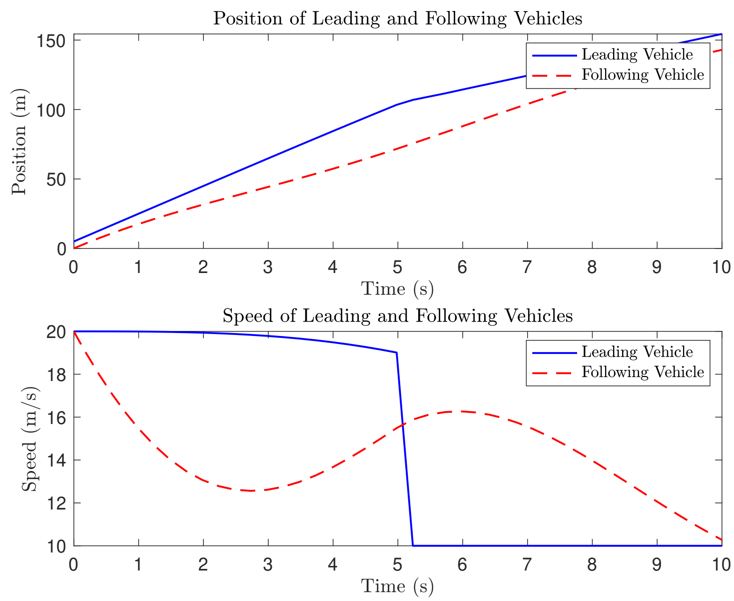
(b) Suppose a leader-and-follower system with two vehicles, speed profile of the leading vehicle to be represented as follows:

$$v_l(t) = \begin{cases} 20 - (\frac{t}{5})^3 & 0 < t \le 5 \\ 10 & t > 5 \end{cases} \tag{1}$$

Initial conditions for the leading vehicle are given as $x_l(0) = 5,\ v_l(0) = 20$, and initial conditions for the following vehicle are given as $x_f(0) = 0,\ v_f(0) = 20$. In this case, unit for the space is meters (m), time is seconds and speed is m/s. Use the given parameters for $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$. Then, simulate the system using Matlab's `ode45` solver and plot the simulation results and produce your code.

```
function car_following_simulation
    alpha_1 = -0.253;
    alpha_2 = 0.23;
    alpha_3 = 0.07;
    alpha_4 = -1.61;

    % Initial conditions
    x1_0 = 5;
```




```
    xf_0 = 0;
    vf_0 = 20;

    tspan = [0 10];

    init_conditions = [x1_0; vf_0; xf_0];

    [t, z] = ode45(@(t,z) system_dynamics(t, z, alpha_1, alpha_2, alpha_3, alpha_4), tspan, init_conditions);

    x1 = z(:, 1);
    vf = z(:, 2);
    xf = z(:, 3);

figure;
subplot(2, 1, 1);
plot(t, x1, 'b-', 'LineWidth', 1.25); % Leading vehicle position plot
hold on;
plot(t, xf, 'r--', 'LineWidth', 1.25); % Following vehicle position plot
hold off;
xlabel('Time (s)', 'Interpreter', 'latex');
ylabel('Position (m)', 'Interpreter', 'latex');
title('Position of Leading and Following Vehicles', 'Interpreter', 'latex');
legend('Leading Vehicle', 'Following Vehicle', 'Interpreter', 'latex');

    set(gcf, 'Color', 'w');
    set(gca, 'FontSize', 12);

subplot(2, 1, 2);
plot(t, arrayfun(@(t) v1(t), t), 'b-', 'LineWidth', 1.25); % Leading vehicle speed plot
```

```
hold on;
plot(t, vf, 'r--', 'LineWidth', 1.25); % Following vehicle speed plot
hold off;
xlabel('Time (s)', 'Interpreter', 'latex');
ylabel('Speed (m/s)', 'Interpreter', 'latex');
title('Speed of Leading and Following Vehicles', 'Interpreter', 'latex');
legend('Leading Vehicle', 'Following Vehicle', 'Interpreter', 'latex');

    set(gcf, 'Color', 'w');
    set(gca, 'FontSize', 12);
end

function dzdt = system_dynamics(t, z, alpha_1, alpha_2, alpha_3, alpha_4)
    x1 = z(1);
    vf = z(2);
    xf = z(3);

    v1_speed = v1(t);
    delta_x = x1 - xf;
    delta_v = v1_speed - vf;

    dx1_dt = v1_speed;

    dvf_dt = alpha_1 * vf + alpha_2 * delta_x + alpha_3 * delta_v + alpha_4;
    dxf_dt = vf;

    dzdt = [dx1_dt; dvf_dt; dxf_dt];
end

function speed = v1(t)
    if t <= 5
        speed = 20 - (t/5)^3;
    else
        speed = 10;
    end
end
```

2. Modeling macroscopic traffic dynamics:

In this problem you will simulate and analyze the behavior of traffic on a particular highway. The stretched highway spans 5 kilometers. The traffic on this highway is modeled using the LWR model along with the Greenshields fundamental diagram. As such, the highway is divided into several segments with uniform length where each segment is 500 meters long. The free flow speed of the traffic is limited to 70 miles per hour (or equivalently 112.654 kilometers per hour) while the maximum density of the traffic is approximately 53 vehicles per kilometer. The stretched highway is equipped with on-ramps to let vehicles from surrounding roads merge into the highway. The ramps are located on segments number 2, 4, 6, 8. For simplicity, one segment is limited to have one on-ramp at most. The questions for this problem are as follows:

(a) Assuming that the highway is in the uncongested mode (as discussed in class), please express the traffic dynamics in the form of

$$\dot{x}(t) = Ax(t) + f(x) + Bu(t)$$

where the state x collects all the traffic densities on all segments located on the stretched highway as well as the on-ramps and the input u collects the traffic flow entering the highway from the upstream segment as well as flows entering on-ramps. In this case, what do matrix A, matrix B, and function $f(\cdot)$ look like?

(b) Suppose that the traffic flow entering the highway from the upstream segment is 0.09 veh/s and the flows entering on-ramps located on segments number 2, 4, 6, 8 are 0.025 veh/s, 0.055 veh/s, 0.045 veh/s, and 0.035 veh/s respectively. The aforementioned traffic flows are assumed to be constant at all times. The traffic densities at $t = 0$ sec on the stretched highway (from upstream to downstream segments) are 0.023 veh/m, 0.01 veh/m, 0.006 veh/m, 0.017 veh/m, 0.015 veh/m, 0.012 veh/m, 0.015 veh/m, 0.009 veh/m, 0.012 veh/m, 0.013 veh/m, while the densities on the on-ramps are 0.02 veh/m, 0.012 veh/m, 0.019 veh/m, 0.05 veh/m. Given the above set-up, find at least one equilibrium point for the system numerically. You can use MATLAB function `fsolve` to compute the equilibrium point(s). Show your MATLAB codes.

(c) Using the traffic parameters, flows, and inital conditions given previously, plot the evolution of traffic densities on all segments (including the ramps) starting from $t = 0$ sec to $t = 500$ sec. You can use MATLAB's ODE solver `ode23s` to simulate the system. Do the traffic densities converge to one of the equilibrium point you obtained previously? Show your MATLAB codes.

```
function traffic_simulation
    l = 500;
    vf = 112.654 / 3.6;
    N = 10;
    fin = 0.09;
    NI = 5;
    f_ramps = [0, 0.025, 0, 0.045, 0, 0.035, 0, 0.055, 0, 0];
    f_out = [0.023, 0.01, 0.006, 0.017, 0.015, 0.009, 0.012, 0.013, 0.02, 0.019];

    A1 = -vf/l * eye(N);
    for i = 2:N
        A1(i, i-1) = vf/l;
    end

    A2 = zeros(N, N+NI);
    for i = 1:NI
        A2(i, N+i) = vf/l;
        A2(N+i, i) = -alpha(i) * vf/l;
```

```
    end

    A3 = zeros(N+NI, N+NI);
    for i = 1:NI
        A3(N+i, N+i) = -vf/l;
        A3(i, N+i) = alpha(i) * vf/l;
    end

    B_u = zeros(N, 1 + NI + NO);

    B_u(1, 1) = 1/l;
    for i = 1:NI
        B_u(N+i, 1+i) = 1/l;
    end
    for i = 1:NO
        B_u(N+NI+i, 1+NI+i) = -1/l;
    end

    x0 = zeros(N, 1);

    function dxdt = traffic_ode(t, x)
        dxdt = A*x + f(x) + B_u;
    end

    function dx = f(x)
        dx = zeros(N, 1);
        for i = 2:N
            dx(i) = -f_out(i-1) + f_ramps(i);
        end
    end

    options = optimoptions('fsolve', 'Display', 'iter', 'TolFun', 1e-8, 'MaxFunctionEvaluations', 1000);
    equilibrium = fsolve(@(x) traffic_ode(0, x), x0, options);

    disp('Equilibrium point:');
    disp(equilibrium);

    [t, x] = ode23s(@(t, x) traffic_ode(t, x), [0, 500], x0);

    figure;
    plot(t, x);
    xlabel('Time (s)');
    ylabel('Traffic density (vehicles/km)');
    title('Traffic density evolution over time');
    legend(arrayfun(@(i) sprintf('Segment %d', i), 1:N, 'UniformOutput', false));
end
```

3. Ring road experiment (optional/bonus):

   Go to `https://traffic-simulation.de/ring.html`. Follow the introduction on how to use the data in class. Create a jam first and propose at least 2 methods to mitigate the congestion. Show your space-time diagram and describe what action you did.