HOMEWORK # 5
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Due date of this homework is: October 25th at 23:59:59.

Problem 0 — Simulating Dynamic Systems

1. Go through the following links:
   (a) State space on MATLAB: http://www.mathworks.com/videos/state-space-models-part-1-creation-and-analysis-100815.html
   (b) Another way of simulating state space using ODE solver: https://www.mathworks.com/matlabcentral/answers/146782-solve-state-space-equation-by-ode45

2. In this link http://academic.csuohio.edu/richter_h/courses/mce371/mce371_5.pdf, you'll find a quick introduction to state space and its implementation on MATLAB, similar to the one above.
   (a) Go through Pages 3–14 of this PDF presentation. Make sure that you understand the details involved.
   (b) You are now given the following dynamical system (identical to the one given in the PDF):
   $$2y^{(4)}(t) + 0.9y^{(3)}(t) + 45.1\ddot{y}(t) + 10\dot{y}(t) + 250y(t) = 250u(t),$$
   where $y(t)$ and $u(t)$ are the output and input to the system. Derive **two different** state space representations, i.e., **obtain two sets** of state-space matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ for this fourth order ODE. Two different state space realizations can be the controllable and observable canonical forms, which were given by the forms derived in the slides (Slides 44-48).
   (c) For each set of the derived matrices, and given what you learned about the ODE solvers on MATLAB, simulate the dynamics of this system assuming that the input $u(t)$ is a unit step function ($u(t) = 1$). Consider that the time horizon is equal to 2 seconds. You'll have to plot the states of the system with respect to time, as well as the output $y(t)$. You can assume any set of initial conditions (do not change the initial conditions for the two state space representations).
   (d) Is there a difference between the output and the states for the two state-space representations? Why/Why Not? Explain your answer.

Problem 1 — Solution of a CT System [Slide 88]

Given a CTLTI model,

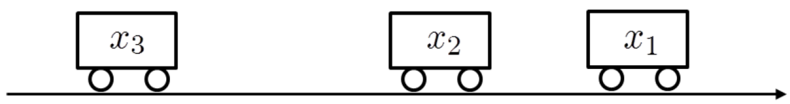
$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = T \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} T^{-1}, B = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix}, a \neq 0, b \neq 0.$$

1. Determine e^{At}.
2. Find $e^{A(t-\tau)}B$.
3. Given that $u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{bt}$ and $x(2) = T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $x(0)$.

Problem 2 — Solution of a CT System [Slide 88]



Consider three cars moving on the same lane, whose initial locations at time $t = 0$ are $x_1(0) = x_2(0) = x_3(0) = 0$. The above figure exemplifies the movement of cars in 1-D. Suppose the goal is for all three cars to meet at the same location (it does not matter where this meet-up location is). To achieve this goal, the following system dynamics can be designed, where $u(t)$ is an input control for the leading car:

$$\dot{x}_1(t) = x_2(t) - x_1(t) + u(t) \quad (1)$$

$$\dot{x}_2(t) = \frac{x_1(t) + x_3(t)}{2} - x_2(t) \quad (2)$$

$$\dot{x}_3(t) = x_2(t) - x_3(t) \quad (3)$$

In other words, the leading and trailing cars will both move toward the middle car instantaneously; while the middle car will move towards the center of the leading and the trailing cars. The derivative of each individual state is obviously the velocity.

1. Represent the above dynamics of the three controls as an LTI dynamical system:

   $$\dot{x}(t) = Ax(t) + Bu(t),$$

   where A, B are matrices that you should determine.
2. Find e^{At} for all $t \in \mathbb{R}$.
3. Suppose $u(t) = 1, t \geq 0$. Find the expression of $x(t)$ for $t \geq 0$.
4. Describe the steady-state behaviors of $x_i(t), i = 1, 2, 3$. Your description must have physical, applied meaning.
5. Simulate the above system via the ODE45 solver on Matlab. Test out different initial conditions as well as different control inputs and observe and plot the outputs.

Problem 3 — State-Feedback Controller [Slides 171–175]

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t),\ A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Assume that a linear state-feedback controller of this form

$$u(t) = Kx(t) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix} x(t)$$

is used as a control input.

1. Find $A + BK$ in terms of $k_1, \ldots, k_8$.
2. Find K such that $A + BK$ is block-diagonal (i.e., formed by two blocks of 2-by-2 matrices on the diagonal and zeros elsewhere.) and the first block has eigenvalues (2,3) and the second block has eigenvalues (0,1).

Problem 4 — Linear Systems Properties [Slides 153—192]

Determine whether the following system is controllable, observable, detectable, and stabilizable. You have to justify your answer.

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -4 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t).$$

Problem 5 — Linearization of Nonlinear Systems [Slides 133–136]

1. Obtain the linearized state space representation of the following nonlinear system around $x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix}$ and $u_e = u^*$ (i.e., these quantities are known and given):

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\sin(x_2(t)) + x_2(t)u(t) \\ \dot{x}_2(t) &= x_1(t)e^{-x_2(t)} + u^2(t) \\ y(t) &= 2x_1(t)x_2(t) + x_2^2(t). \end{aligned}$$

2. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t)(x_1^2(t) - 1) \\ \dot{x}_2(t) &= x_2^2(t) + x_1(t) - 3 \end{aligned}$$

   (a) Find all the equilibrium points of the nonlinear system.

   (b) Determine the stability of the system around each equilibrium point, if possible. You can verify your solutions by plotting phase portraits on MATLAB.

Problem 6 — Observer Design [Slides 196–200]

A dynamical CTLTI system is characterized by $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} -0.5 & 1 \end{bmatrix}$.

1. Find a linear state-observer gain $L = [l_1 \ l_2]^\top$ such that the poles of the estimation error are -5 and -7.

2. Can you place both poles at -6? If yes, what is the corresponding observer gain?

3. Simulate the above observer/state estimator for various initial conditions and showcase how the state estimation error converges to zero with various observer gains.

Problem 7 — Observer-Based Controller [Slides 202–206]

Design an OBC (i.e., $u(t) = -K\hat{x}(t)$) for the following SISO system

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} -7 & 0 \end{bmatrix} x(t).$$

You can place the closed-loop system eigenvalues wherever you want. Simulate the OBC-controlled system via the ODE45 solver for any initial condition of your choice. Also, simulate the uncontrolled system and note the difference. What do we learn?

Problem 8 — Discretization of Continuous Systems [Slides 101–109]

Solve the following related problems via MATLAB.

1. Using MATLAB, generate a random LTI dynamical system of 5 states, 3 control inputs, and 3 outputs. You can use `randn` or `rand` commands on MATLAB, or you can simply come up with your own random state-space matrices as we did in class. **Ensure that the systems are stable. That means you have to generate random matrices that yield a stable system, i.e., eigenvalues of matrix A are in the LHP.**

2. Simulate the system given that the inputs are $\text{square}(t), \sin(t), \cos(t)$ and the initial conditions for the system are identically zero. Simulate the system using the ode45 solver.

3. Now, apply the two discretization methods we discussed in class with variable sampling periods. For example trying sampling periods of $T = 0.01, 0.1$, and 5 seconds. Discuss the outcome that you get between the accurate ODE solver and two discretization methods. Is the discretization always accurate? When does it fail (if it does)? Include your code, plots, and a thorough analysis of the results.

4. Plot the norm of the discretization error $||x_{\text{ode}}(t) - x_{\text{discretization}}(t)||$ for the two methods for each of the sampling periods given in the previous problem.