# Module 5
# Intro to Drinking Water Distribution Networks

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**CE 4240/5240 — Intro to Infrastructure Systems Engineering**

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# Outline

This presentation covers the dynamic models of the drinking water distribution networks (WDNs):

1. WDN Research; Motives, Challenges, and Objectives.
2. WDN: Hydraulic Dynamics.
3. WDN: Quality Dynamics.
4. Control-oriented WDN Research Topics.

# ***Part I**:*
# *WDN Research, Challenges, and Objectives*

# Part I: Motivation for WDN Research

## Water Distribution Networks

- **Purpose**: deliver water to consumer with appropriate quality, quantity and pressure, but
  - ❑ Limited natural water sources
  - ❑ Population is increasing rapidly
  - ❑ Complexity/uncertainty in WDNs


To effectively deliver water, need to study hydraulics & quality dynamics

- **Water Flow problem**
  - State Estimation
  - Optimal Control
- **Water Quality Model**
  - Water Quality Control
  - Sensor Placement

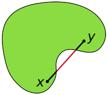
# Part I: Technical challenges

## Flow Direction

The flow direction in a pipe varies in network

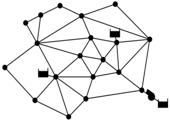
## Non-convexity Non-linearity

Pressure models: non-differentiable, non-convex, nonlinear




## Topology

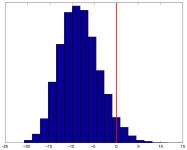
Grid, ring, or tree



# Part I: Technical challenges

## Varying Uncertainty

Distribution-free uncertainty (demand, pipe parameters, sensor noise)



## No analytical solution

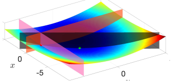
The quality modeling is presented by 1-D advection-reaction (A-R) equation (PDE)

## Coupled problem

Need to solve the coupled

Optimal Control in Hydraulic & Quality

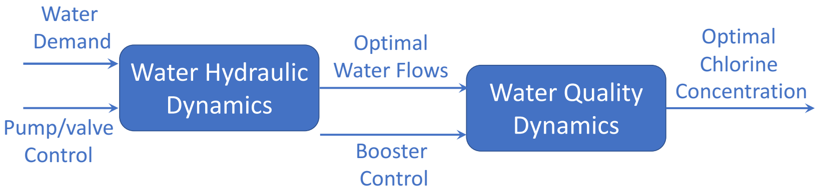
problem



# Part I: Objectives

**Objective:** Simulate hydraulic and quality dynamics, given:

- Network topology, parameters, and demands
- Control actions from pumps (that control hydraulics) & booster stations (that injectchlorine to decontaminate the network)



# Part I: Objective for this class

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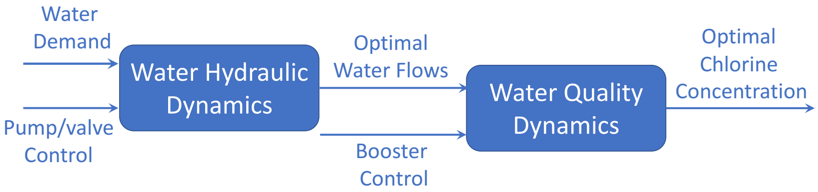
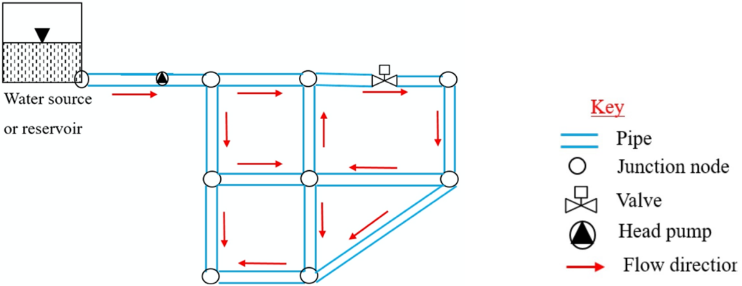
- *Network topology, parameters, and demands*
- *Control actions from pumps (that control hydraulics) & booster stations (that inject chlorine to decontaminate the network)*



***Part II**:*
*Water Network Hydraulic Dynamics*

# Part II: Hydraulics

## Components




Components in Water Distribution Networks (WDNs):

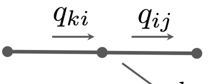
- Nodes: junctions, tanks or reservoirs
- Links: pumps, valves, or pipes
  - Fixed or variable speed pumps
  - Various types of control valves: PRV (pressure reducing valves), FCV (flow control valves), etc...

In water systems, the head is denoted by $h$ (in feet); the flow rate is denoted by $q$ (in gallon per minute or cubic feet per second).

# Part II: Hydraulics

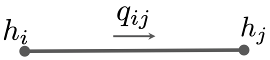
## Mass/Energy Balance equation in each component

Mass balance (linear) Energy balance (nonlinear)



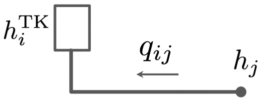

**(Junction)**

$$q_{ki} - q_{ij} = d_i$$



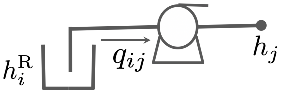

**(Pipe)**

$$h_i - h_j = \Delta h_{ij}^{\mathrm{P}}$$




**(Tank)**

$$h_i^{\mathrm{TK}}(k+1) = h_i^{\mathrm{TK}}(k) + \alpha_i \; q_{ij}(k)$$



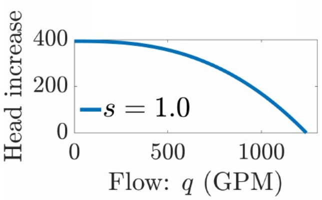

**(Reservoir and Pump)**

$$h_j - h_i^R = \Delta h_{ij}^{\mathrm{M}}$$

# Part II: Hydraulics: Pump Characteristic Curve

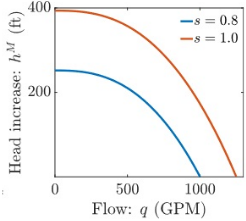
## Pump Characteristic Curve

**Fixed Speed Pump:**




$$\Delta h_{ij}^{\mathrm{M}}(k) = s^2(h_0 - r(q(k)/s)^{\nu})$$

**Variable Speed Pump:**



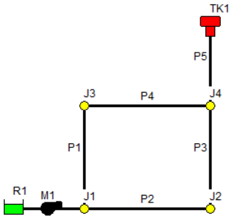

$$\Delta h_{ij}^{\mathrm{M}}(k) = s^2(k)(h_0 - r(q(k)/s(k))^{\nu})$$

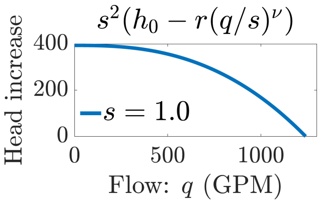
# Part II: Hydraulics

## Mass/Energy Balance

| Component | Hydraulic Modeling | Description |
|---|---|---|
| Junction | $\sum q^{\mathrm{in}}(k) - \sum q^{\mathrm{out}}(k) = d(k)$ | Inflow – Outflow=Demand |
| Tank | $h^{\mathrm{TK}}(k+1) = h^{\mathrm{TK}}(k) + \alpha(\text{net flow})$ | Tank Volume/Head Dynamics |
| Pipe | $\Delta h^{\mathrm{P}}(k) = Rq(k)\,\lvert q(k)\rvert^{\mu-1}$ | Head loss in pipes due to friction |
| Pump | $\Delta h^{\mathrm{M}}(k) = h_0 - rq^{\nu}(k)$ | Head gain through pump control |

# Part II: Hydraulics – An Example




**Suppose that we have parameters as below**:

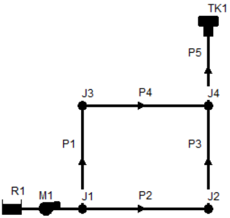
- Reservoir R1 has fixed head (pressure), that is $h_{\mathrm{R1}} = 700$ ft
- Pump M1 is fully on ($s = 1$), and the head increase curve is shown in the figure (right), where speed $h_0 = 389.72$ ft, $r = 3.746 \times 10^{-6}$, $\nu = 2.59$. Water is pumped into Junction J1.
- Junction J2 consumes $200$ GPM, that is, $d_2 = 200$ GPM. Similarly, $d_3 = 100$ GPM and $d_4 = 150$ GPM.
- Resistance coefficient $R = 1.144 \times 10^{-5}$ in all pipes, and flow exponent $\mu = 1.852$
- The initial head in Tank 3 is $908$ ft, that is $h_{\mathrm{TK1}}(0) = 908$ ft, and the head change rate $\alpha = 6.8 \times 10^{-5}$ feet $\cdot$ hour per gallon.
- Each time-step is equivalent to one hour.

**Operator task**:

- **Compute** the water head (pressure) in Tank TK1 after 1 hour?

# Part II: Hydraulics (Example Cont'd)

First, assume flow directions and formulate accordingly. It's important to note that we assume these flow directions to establish consistent equations for both mass and energy balance. By solving these equations, negative flow values indicate that the actual flow direction is opposite to our initial assumptions. If your initial flow direction assumption was wrong, the results will have different signs—most importantly, being consistent is what's important.



# Part II: Hydraulics (Example Cont'd)

**Mass Balance in Junctions:**

$$\begin{aligned} \text{J1}: &\ q_{\text{M1}} - q_{\text{P1}} - q_{\text{P2}} = 0, \\ \text{J2}: &\ q_{\text{P2}} - q_{\text{P3}} = d_2, \\ \text{J3}: &\ q_{\text{P1}} - q_{\text{P4}} = d_3, \\ \text{J4}: &\ q_{\text{P3}} + q_{\text{P4}} - q_{\text{P5}} = d_4. \end{aligned}$$

**Energy Balance through Pump:**

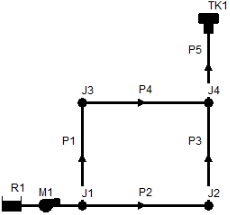
$$h_{\text{J1}} - h_{\text{R1}} = h_{\text{J1}} - 700 = 389.72 - 3.746 \times 10^{-6} \times q_{\text{M1}}^{2.59}.$$

**Energy Balance through Pipes:**

$$\begin{aligned} \text{P1}: &\ h_{\text{J1}} - h_{\text{J3}} = 1.144 \times 10^{-5} \times q_{\text{P1}} \times |q_{\text{P1}}|^{0.852}, \\ \text{P2}: &\ h_{\text{J1}} - h_{\text{J2}} = 1.144 \times 10^{-5} \times q_{\text{P2}} \times |q_{\text{P2}}|^{0.852}, \\ \text{P3}: &\ h_{\text{J2}} - h_{\text{J4}} = 1.144 \times 10^{-5} \times q_{\text{P3}} \times |q_{\text{P3}}|^{0.852}, \\ \text{P4}: &\ h_{\text{J3}} - h_{\text{J4}} = 1.144 \times 10^{-5} \times q_{\text{P4}} \times |q_{\text{P4}}|^{0.852}, \\ \text{P5}: &\ h_{\text{J4}} - h_{\text{TK1}} = 908 - h_{\text{J4}} = 1.144 \times 10^{-5} \times q_{\text{P5}} \times |q_{\text{P5}}|^{0.852}. \end{aligned}$$

**Mass Balance in Tank:**

$$h_{\text{TK1}}(1) = 908 + 6.8 \times 10^{-5} \times 60 \times q_{\text{P5}}.$$



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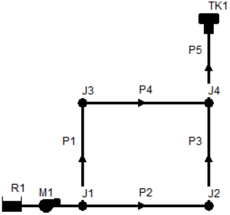

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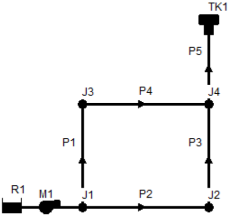


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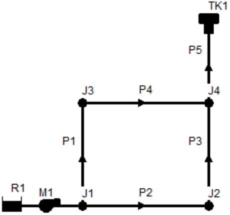


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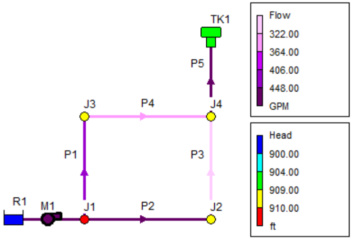
# Part II: Hydraulics

## Example

By solving these equations, we get:

$$q_{\text{M1}} = 922.98, \quad q_{\text{P1}} = 440.64, \quad q_{\text{P2}} = 482.34,$$
$$q_{\text{P3}} = 282.34, \quad q_{\text{P4}} = 340.64, \quad q_{\text{P5}} = 472.98,$$
$$h_{\text{J1}} = 910.49, \quad h_{\text{J2}} = 909.43, \quad h_{\text{J3}} = 909.59,$$
$$h_{\text{J4}} = 909.03,$$
$$h_{\text{TK1}}(1) = 909.93$$




- But how do you solve this nonlinear system of equations?
- Well, this problem is called the **Water Flow Problem**
- For small system, we can solve this problem on MATLAB using $\text{fsolve}$
- For larger systems? Formulate feasibility problem:

$$\min_{q,h} \; 0$$
$$\text{subject to} \;\; \text{Mass Balance Equations and Energy Balance Equations}$$

- What are the variables? Is the problem convex? Can use Newton's Method?

# Part II: Hydraulics — State-Space Representation

## Notation

| *Symbol* | *Description* |
|---|---|
| $\boldsymbol{h}^{\mathrm{J}}$, $\boldsymbol{h}^{\mathrm{R}}$, $\boldsymbol{h}^{\mathrm{TK}}$ | Heads at junctions, reservoirs, tanks |
| $\boldsymbol{q}^{\mathrm{P}}$,$\boldsymbol{q}^{\mathrm{M}}$,$\boldsymbol{q}^{\mathrm{W}}$ | Flows in pipes, pumps, valves |
| $\boldsymbol{d}$ | Demands at junctions |

## Differential Algebraic Equations (DAE)

**Tank Dynamic:** $\boldsymbol{h}^{\mathrm{TK}}(k+1) = \boldsymbol{h}^{\mathrm{TK}}(k) + \boldsymbol{E}^{\mathrm{TK}}\boldsymbol{q}^{\mathrm{P}}(k)$

**Mass Balance:** $\boldsymbol{d}(k) = \boldsymbol{E}_q\boldsymbol{q}(k)$

**Energy Balance:** $\boldsymbol{h}(k) = \boldsymbol{E}_h\,\boldsymbol{\Phi}(\boldsymbol{x}(k))$

- Let $\boldsymbol{x} = \{\boldsymbol{h}, \boldsymbol{q}\}$ collect all variables, $\boldsymbol{\Phi} = \{\boldsymbol{\Phi}^{\mathrm{P}}, \boldsymbol{\Phi}^{\mathrm{M}}\}$ collect nonlinear head loss/increase equations, then hydraulics can be written as nonlinear difference algebraic equations (DAE):

$$\boldsymbol{E}\boldsymbol{x}(k+1) = \boldsymbol{A}_x\boldsymbol{x}(k) + \boldsymbol{E}_d\boldsymbol{d}(k) + \boldsymbol{E}_\Phi\boldsymbol{\Phi}(\boldsymbol{x}(k))$$

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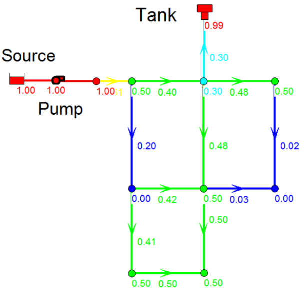
**Is $E$ singular or non-singular?**
**And what does that imply?**

# ***Part III**:*
# *Water Quality Dynamics*

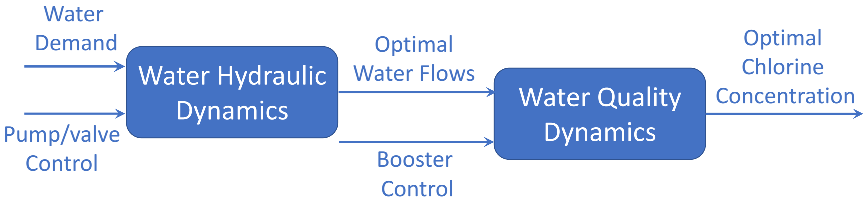
# Part III: Water Quality Background




Facts in Water Distribution Networks (WDNs):

- Chlorine disinfection prevents the spread of waterborne diseases
- When quality is bad, cities issue boil water mandates
- Water utilities in the US are required to preserve detectable chlorine residual (with concentration of 0.2 to 4 milligram per liter) throughout WDNs under the Surface Water Treatment Rule (SWTR)
- Booster stations are used to inject chlorine to WDNs (to simplify, we assume a booster station is installed at the water source)

# Part III: Water Quality Dynamics Background




- Recall that the hydraulic dynamics are derived in Part II
- To control water quality, all we need is the quality dynamics. But which water quality aspect we focus on?
- We focus on chlorine (i.e., the disinfectant) and trace its evolution throughout the network's components.
- Chlorine is responsible to keep the water pathogen-free by working as safeguard against bacteria and microbial contamination.
- Chlorine is a great water quality proxy as it is easy to measure and monitor by water quality sensors.

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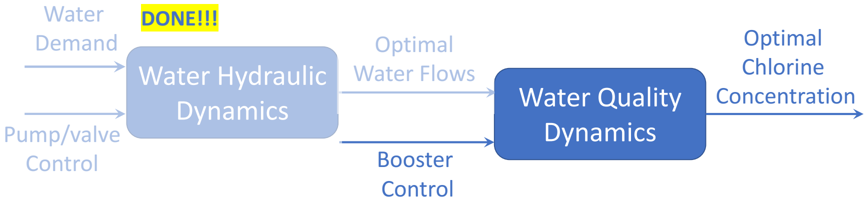


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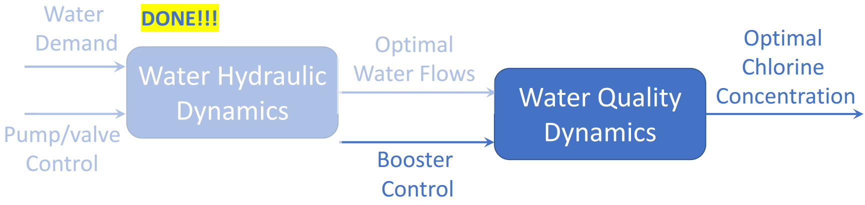


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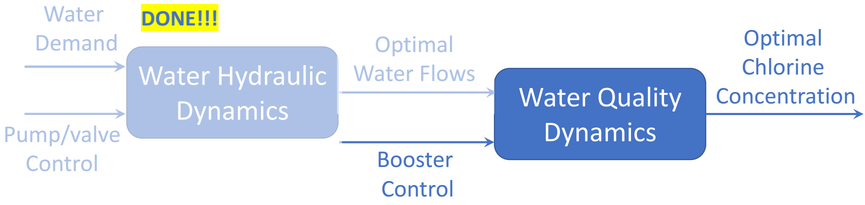
# Part III: Water Quality Dynamics Background




- Recall that the hydraulic dynamics are derived in Part II
- To control water quality, all we need is the quality dynamics. But which water quality aspect we focus on?
- We focus on chlorine (i.e., the disinfectant) and trace its evolution throughout the network's components.
- Chlorine is responsible to keep the water pathogen-free by working as safeguard against bacteria and microbial contamination.
- Chlorine is a great water quality proxy as it is easy to measure and monitor by water quality sensors.

# Part III: Water Quality Dynamics

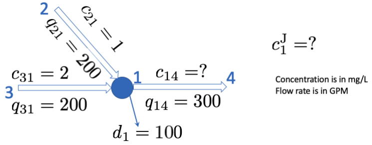
- Chlorine concentrations evolution is covered by the conservation of mass law, transport, decay, and reaction models.
- Chlorine concentrations $c$ are typically measured in mg/L. That is, if water is flowing with a rate of $q$ (GPM) then $cq$ is the mass rate of chlorine.
- By applying the conservation of mass, all input mass rates at a component are mixed (with the storage water if any) and the output rates have the resultant concentrations.

# Part III: Water Quality Dynamics

## Component — Junction

- Conservation of mass is applied at junctions to determine the effect of combing flow with different chlorine concentrations
- At junctions, the water quality (e.g., chlorine concentrations) changes due to the dilution and injections.

To simplify, let us only consider dilution through this example




According to mass balance, we know that $c_{21}q_{21} + c_{31}q_{31} = c_1^J q_{14} + c_1^J d_1$. Hence,

$$c_1^J = \frac{c_{21}q_{21} + c_{31}q_{31}}{d_1 + q_{14}} = \frac{1 \times 200 + 2 \times 200}{100 + 300} = 1.25\text{mg/L}$$

# Part III: Water Quality Dynamics

## Component — Junction

- Conservation of mass is applied at junctions to determine the effect of combing flow with different chlorine concentrations
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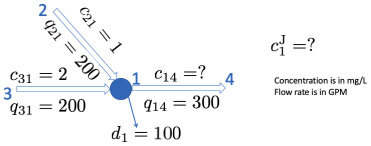


According to mass balance, we know that $c_{21}q_{21} + c_{31}q_{31} = c_1^{\mathrm{J}}q_{14} + c_1^{\mathrm{J}}d_1$. Hence,

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# Part III: Water Quality Dynamics

## Component — Junction

In general, we have

$$c_i^{\mathrm{J}} = \frac{\sum_{ki \in L_{\mathrm{in}}} c_{ki} q_{ki}}{d + \sum_{ij \in L_{\mathrm{out}}} q_{ij}} = \sum \mathrm{coeff}_{ki} \times c_{ki}$$

How about considering injection at a junction?

$$c_i^{\mathrm{J}} = \frac{\sum_{ki \in L_{\mathrm{in}}} c_{ki} q_{ki} + c^{\mathrm{Booster}} q^{\mathrm{Booster}}}{d + \sum_{ij \in L_{\mathrm{out}}} q_{ij}} = \sum \mathrm{coeff}_{ki} \times c_{ki} + \sum \mathrm{coeff}_n \times c_n^{\mathrm{Booster}}$$

$L_{\mathrm{in}}$ are set of links flowing into Junction $i$;
$L_{\mathrm{out}}$ are set of links extracting flow from Junction $i$.

# Part III: Water Quality Dynamics

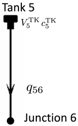
## Component — Tank

Similar to junctions,

- Conservation of mass is applied in tanks
- In tanks, chlorine concentrations change due to the dilution (inflow and outflow) and injection

Let us only consider the outflow first

$$\underbrace{V_5^{\mathrm{TK}}(t+\Delta t)c_5^{\mathrm{TK}}(t+\Delta t)}_{m(t+\Delta t)} = \underbrace{V_5^{\mathrm{TK}}(t)c_5^{\mathrm{TK}}(t)}_{m(t)} - \underbrace{q_{56}(t)\Delta t c_5^{\mathrm{TK}}(t)}_{\text{mass flow out}}$$




Hence,

$$c_5^{\mathrm{TK}}(t+\Delta t) = \frac{V_5^{\mathrm{TK}}(t) - q_{56}(t)\Delta t}{V_5^{\mathrm{TK}}(t+\Delta t)} c_5^{\mathrm{TK}}(t) = \mathrm{coeff}_5 \times c_5^{\mathrm{TK}}(t)$$

How about considering injection at the tank?

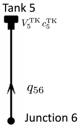
$$c_5^{\mathrm{TK}}(t+\Delta t) = \frac{V_5^{\mathrm{TK}}(t) - q_{56}(t)\Delta t}{V_5^{\mathrm{TK}}(t+\Delta t)} c_5^{\mathrm{TK}}(t) + \frac{c^{\mathrm{Booster}}(t)q^{\mathrm{Booster}}(t)\Delta t}{V_5^{\mathrm{TK}}(t+\Delta t)}$$
$$= \mathrm{coeff}_5 \times c_5^{\mathrm{TK}}(t) + \mathrm{coeff}^{\mathrm{Booster}} c^{\mathrm{Booster}}(t)$$

# Part III: Water Quality Dynamics

## Component — Tank

Now, let us consider inflow scenario,

$$\underbrace{V_5^{\mathrm{TK}}(t+\Delta t)c_5^{\mathrm{TK}}(t+\Delta t)}_{m(t+\Delta t)} = \underbrace{V_5^{\mathrm{TK}}(t)c_5^{\mathrm{TK}}(t)}_{m(t)} + \underbrace{q_{56}(t)\Delta t c_{65}^{\mathrm{P}}(s_L,t)}_{\text{mass flow in}}$$




Hence,

$$c_5^{\mathrm{TK}}(t+\Delta t) = \frac{V_5^{\mathrm{TK}}(t)}{V_5^{\mathrm{TK}}(t+\Delta t)}c_5^{\mathrm{TK}}(t) + \frac{q_{56}(t)\Delta t}{V_5^{\mathrm{TK}}(t+\Delta t)}c_{65}^{\mathrm{P}}(s_L,t)(t)$$
$$= \mathrm{coeff}_5 \times c_5^{\mathrm{TK}}(t) + \mathrm{coeff}_{65} \times c_{65}^{\mathrm{P}}(s_L,t),$$

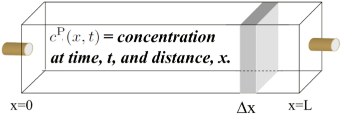
How about considering injection at the tank?

$$c_5^{\mathrm{TK}}(t+\Delta t) = \mathrm{coeff}_5 \times c_5^{\mathrm{TK}}(t) + \mathrm{coeff}_{65} \times c_{65}^{\mathrm{P}}(s_L,t) + \mathrm{coeff}^{\mathrm{Booster}} c^{\mathrm{Booster}}(t)$$

# Part III: Water Quality Dynamics

## Component — pipe

The chlorine transport and reaction modeling in Pipe $ij$ can be described by means of the 1-D advection-reaction (A-R) equation




Partial Differential Equations (PDE) ⟵ $$\partial_t c^{\mathrm{P}} = -u_{ij}(t)\partial_x c^{\mathrm{P}} + r^{\mathrm{P}}_{ij}(c^{\mathrm{P}})$$

where $u_{ij}$ is the flow velocity, $r^{\mathrm{P}}_{ij}(c^{\mathrm{P}}) = k_{ij}(c^{\mathrm{P}})^n$ is the *n*-th order reaction expression.

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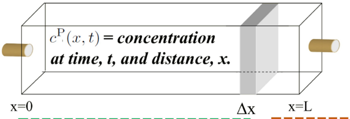


Partial Differential Equations (PDE) ⟵

$$\partial_t c^{\mathrm{P}} = -u_{ij}(t)\partial_x c^{\mathrm{P}} + r_{ij}^{\mathrm{P}}(c^{\mathrm{P}})$$

$\partial_t c^{\mathrm{P}} \rightarrow \frac{\partial c^{\mathrm{P}}}{\partial t}$, $\partial_x c^{\mathrm{P}} \rightarrow \frac{\partial c^{\mathrm{P}}}{\partial x}$: Advection/Transport

$r_{ij}^{\mathrm{P}}(c^{\mathrm{P}})$: Reaction ⟶ Chlorine Decay

where $u_{ij}$ is the flow velocity, $r_{ij}^{\mathrm{P}}(c^{\mathrm{P}}) = k_{ij}(c^{\mathrm{P}})^n$ is the *n*-th order reaction expression.

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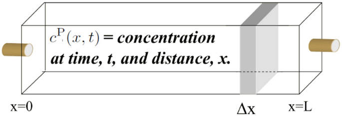



No analytical solution for this PDE.

Partial Differential Equations (PDE) ←

$$\partial_t c^{\mathrm{P}} = -u_{ij}(t)\partial_x c^{\mathrm{P}} + r^{\mathrm{P}}_{ij}(c^{\mathrm{P}})$$

$\partial_t c^{\mathrm{P}} \rightarrow \dfrac{\partial c^{\mathrm{P}}}{\partial t}$, $\partial_x c^{\mathrm{P}} \rightarrow \dfrac{\partial c^{\mathrm{P}}}{\partial x}$: Advection/Transport

$r^{\mathrm{P}}_{ij}(c^{\mathrm{P}})$: Reaction → Chlorine Decay

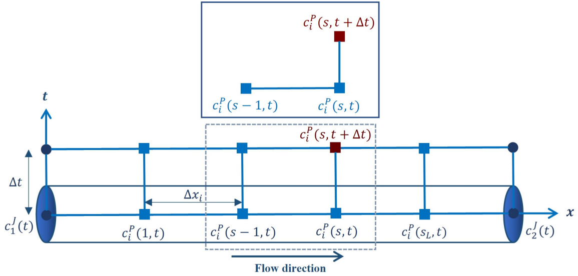
where $u_{ij}$ is the flow velocity, $r^{\mathrm{P}}_{ij}(c^{\mathrm{P}}) = k_{ij}(c^{\mathrm{P}})^n$ is the $n$-th order reaction expression.

**Objective:** Solve this PDE for each pipe in the network

**Approach:** Use numerical discretization over space and time methods

# Part III: Water Quality Dynamics

## Component — pipe

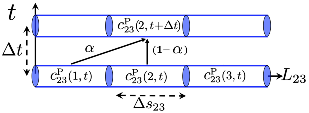



Upwind discretization scheme is one of the numerical methods based on finite differences for the solution of PDE. A pipe is split into many segments using this numerical method.

# Part III: Water Quality Dynamics

## Component — pipe

Let us consider a simple example; Pipe 23 is split into three segments below.



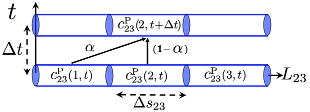

According to the scheme at a specific time of the simulation period, the concentration at each pipe segment depends on its value and the concentration of the upstream segment/node at the previous time-step.

To calculate the concentrations at each segment, a parameter $\alpha$ is defined depending on the **flow velocity** in the pipe $u_{ij}$ and the **number of segments** the pipe is split into.

# Part III: Water Quality Dynamics

## Component — pipe

Pipe 23 is split into three segments as shown below.




By applying the upwind scheme, we obtain

$$c^{\text{P}}_{23}(2, t+\Delta t) = \alpha c^{\text{P}}_{23}(1, t) + (1-\alpha)c^{\text{P}}_{23}(2, t) + r^{\text{P}}_{23}\left(c^{\text{P}}_{23}(s, t)\right)$$

If we consider the first-order decay dynamics, then $r^{\text{P}}_{23}\left(c^{\text{P}}_{23}(s, t)\right) = r_{23} \times c^{\text{P}}_{23}(2, t)$. Hence,

$$c^{\text{P}}_{23}(2, t+\Delta t) = \alpha c^{\text{P}}_{23}(1, t) + (1-\alpha + r_{23})c^{\text{P}}_{23}(2, t),$$

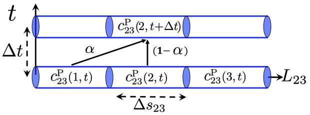
How about the other segments of Pipe 23?

$$c^{\text{P}}_{23}(3, t+\Delta t) = \alpha c^{\text{P}}_{23}(2, t) + (1-\alpha + r_{23})c^{\text{P}}_{23}(3, t)$$
$$c^{\text{P}}_{23}(1, t+\Delta t) = \alpha c^{\text{J}}_{2}(t) + (1-\alpha + r_{23})c^{\text{P}}_{23}(1, t).$$

# Part III: Water Quality Dynamics

## Component — pipe

Pipe 23 is split into three segments as shown below.

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$$c_{23}^{\mathrm{P}}(2, t+\Delta t) = \alpha c_{23}^{\mathrm{P}}(1, t) + (1-\alpha) c_{23}^{\mathrm{P}}(2, t) + r_{23}^{\mathrm{P}}\left(c_{23}^{\mathrm{P}}(s, t)\right)$$

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How about the other segments of Pipe 23?

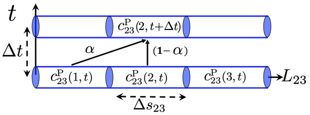
$$c_{23}^{\mathrm{P}}(3, t+\Delta t) = \alpha c_{23}^{\mathrm{P}}(2, t) + (1-\alpha + r_{23}) c_{23}^{\mathrm{P}}(3, t)$$

$$c_{23}^{\mathrm{P}}(1, t+\Delta t) = \alpha c_{2}^{\mathrm{J}}(t) + (1-\alpha + r_{23}) c_{23}^{\mathrm{P}}(1, t).$$

# Part III: Water Quality Dynamics

## Component — pipe

Pipe 23 is split into three segments as shown below.

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$$c^{\mathrm{P}}_{23}(2, t+\Delta t) = \alpha c^{\mathrm{P}}_{23}(1,t) + (1-\alpha) c^{\mathrm{P}}_{23}(2,t) + r^{\mathrm{P}}_{23}\left(c^{\mathrm{P}}_{23}(s,t)\right)$$

If we consider the first-order decay dynamics, then $r^{\mathrm{P}}_{23}\left(c^{\mathrm{P}}_{23}(s,t)\right) = r_{23} \times c^{\mathrm{P}}_{23}(2,t)$. Hence,

$$c^{\mathrm{P}}_{23}(2, t+\Delta t) = \alpha c^{\mathrm{P}}_{23}(1,t) + (1-\alpha+r_{23}) c^{\mathrm{P}}_{23}(2,t),$$

How about the other segments of Pipe 23?

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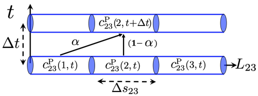
# Part III: Water Quality Dynamics

## Component — pipe




How about the matrix form for Pipe 23?

$$\underbrace{\begin{bmatrix} c_{23}^{\mathrm{P}}(1, t+\Delta t) \\ c_{23}^{\mathrm{P}}(2, t+\Delta t) \\ c_{23}^{\mathrm{P}}(3, t+\Delta t) \end{bmatrix}}_{\boldsymbol{c}_{23}^{\mathrm{P}}(t+\Delta t)} = \underbrace{\begin{bmatrix} 1-\alpha+r_{23} & & \\ \alpha & 1-\alpha+r_{23} & \\ & \alpha & 1-\alpha+r_{23} \end{bmatrix}}_{\boldsymbol{A}_{23}} \underbrace{\begin{bmatrix} c_{23}^{\mathrm{P}}(1, t) \\ c_{23}^{\mathrm{P}}(2, t) \\ c_{23}^{\mathrm{P}}(3, t) \end{bmatrix}}_{\boldsymbol{c}_{23}^{\mathrm{P}}(t)}$$

$$+ \underbrace{\begin{bmatrix} \alpha & & \\ & 0 & \\ & & 0 \end{bmatrix}}_{\boldsymbol{A}_{23}^{\mathrm{J}}} \begin{bmatrix} c_{2}^{\mathrm{J}}(t) \\ 0 \\ c_{3}^{\mathrm{J}}(t) \end{bmatrix}$$

# Part III: Water Quality Dynamics

## Component — pump

Although the hydraulic model of a pump is *complicated*, the quality dynamic of a pump is *super simple*.

Suppose the upstream node of Pump $ij$ is a reservoir (with fixed concentration), and it means the concentration in Pump $ij$ equals the one in Reservoir $i$. In this case,

$$c_{ij}^{\mathrm{M}}(t+\Delta t) = c_i^{\mathrm{R}}(t+\Delta t) = c_i^{\mathrm{R}}(t) = c_{ij}^{\mathrm{M}}(t)$$

Suppose the upstream node of Pump $ij$ is a junction, and we have

$$c_{ij}^{\mathrm{M}}(t+\Delta t) = c_i^{\mathrm{J}}(t+\Delta t)$$

# Part III: Water Quality Dynamics

## WQM — all components

| Component | Water Quality Modeling | Description |
|---|---|---|
| Junction | $\boldsymbol{c}^{\mathrm{J}}(t) = \boldsymbol{A}_{\mathrm{J}}^{\mathrm{L}}(t)\boldsymbol{c}^{\mathrm{L}}(t) + \boldsymbol{B}_{\mathrm{J}}(t)\boldsymbol{c}^{\mathrm{B}}(t)$ | Concentration at junctions = the sum of concentrations from links and boosters |
| Tank | $\boldsymbol{c}^{\mathrm{TK}}(t+\Delta t) = \boldsymbol{A}_{\mathrm{TK}}^{\mathrm{TK}}(t)\boldsymbol{c}^{\mathrm{TK}}(t) + \boldsymbol{A}_{\mathrm{TK}}^{\mathrm{P}}(t)\boldsymbol{c}^{\mathrm{P}}(t) + \boldsymbol{B}_{\mathrm{TK}}(t)\boldsymbol{c}^{\mathrm{B}}(t)$ | Concentration dynamics in tanks |
| Pipe | $\boldsymbol{c}^{\mathrm{P}}(t+\Delta t) = \boldsymbol{A}_{\mathrm{P}}^{\mathrm{P}}(t)\boldsymbol{c}^{\mathrm{P}}(t) + \boldsymbol{A}_{\mathrm{N}}^{\mathrm{P}}(t)\boldsymbol{c}^{\mathrm{J}}(t)$ | Concentration dynamics in all pipes |
| Pump | $\boldsymbol{c}^{\mathrm{M}}(t) = \boldsymbol{E}^{\mathrm{NM}}\boldsymbol{c}^{\mathrm{N}}(t)$ | Concentrations of pumps |

# Part III: Water Quality Dynamics — State-space representation

## Linear Difference Equation (LDE)

### Notation

| *Symbol* | *Description* |
|---|---|
| $\boldsymbol{c}^{\mathrm{J}}$, $\boldsymbol{c}^{\mathrm{TK}}$, $\boldsymbol{c}^{\mathrm{R}}$ | Concentrations at junctions, tanks, reservoirs |
| $\boldsymbol{c}^{\mathrm{N}}$ | $\boldsymbol{c}^{\mathrm{N}} \triangleq \{\boldsymbol{c}^{\mathrm{J}}, \boldsymbol{c}^{\mathrm{TK}}, \boldsymbol{c}^{\mathrm{R}}\}$ collects concentrations at all nodes |
| $\boldsymbol{c}^{\mathrm{P}}$, $\boldsymbol{c}^{\mathrm{M}}$, $\boldsymbol{c}^{\mathrm{W}}$ | Concentrations in pipes, pumps, valves |
| $\boldsymbol{c}^{\mathrm{L}}$ | $\boldsymbol{c}^{\mathrm{L}} \triangleq \{\boldsymbol{c}^{\mathrm{P}}, \boldsymbol{c}^{\mathrm{M}}, \boldsymbol{c}^{\mathrm{W}}\}$ collects concentrations in all links |
| $\boldsymbol{c}^{\mathrm{B}}$, $\boldsymbol{q}^{\mathrm{B}}$ | Booster concentration and flow rate |
| $\boldsymbol{r}^{\mathrm{P}}$, $\boldsymbol{r}^{\mathrm{TK}}$ | Reaction rate for pipes, tanks |
| $k^b$, $k^w$ | Bulk and wall reaction rate constant |

### Lumped variables

$$\boldsymbol{\chi}[t_0] \triangleq \{\boldsymbol{c}^{\mathrm{J}}(t), \boldsymbol{c}^{\mathrm{R}}(t), \boldsymbol{c}^{\mathrm{TK}}(t), \boldsymbol{c}^{\mathrm{P}}(t), \boldsymbol{c}^{\mathrm{M}}(t), \boldsymbol{c}^{\mathrm{W}}(t)\}_{t=t_0}^{t=t_0+N_p}$$

### Constraints

$$\boldsymbol{\chi} \in [\boldsymbol{\chi}^{\min}, \boldsymbol{\chi}^{\max}]$$

The control variable is vector $\boldsymbol{u}(t) \triangleq \boldsymbol{c}^{\mathrm{B}}(t)$ .

### Linear Difference Equation (LDE)

$$\boldsymbol{\chi}(t+\Delta t) = \boldsymbol{A}_{\chi}(t)\boldsymbol{\chi}(t) + \boldsymbol{B}_{u}(t)\boldsymbol{u}(t)$$

The advection-reaction equation (PDE) now is presented as the LDE via the Upwind discretization scheme.

# ***Part IV**:*
# *Control Engineering Water Systems Problems*

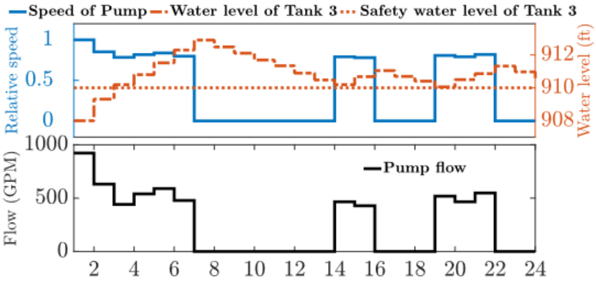
# Hydraulics Control — Pump Scheduling Problem

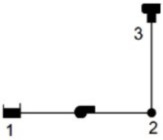
$$\begin{array}{ll} \underset{\Upsilon(t)?}{\text{minimize}} & \Pi_{\mathrm{App}}(\Upsilon(t))\Delta t_{\mathrm{H}} \\ \text{subject to} & \textbf{Hyd-LDAE} \end{array}$$

**Minimize Pump Power Consumption**

**Hydraulic Dynamics: Conservation of Mass and Energy**

**Linearization/Relaxation**



# Water Quality Control — Chlorine Dosage Scheduling Proble

$$\underset{\mathbf{x}(t),\mathbf{u}_1(t)}{\text{minimize}} \quad \mathcal{J}(\mathbf{u}_1(t)) = \mu \sum_{t=1}^{N_p} \mathbf{q}^{\mathrm{B}}(t)^{\top}\mathbf{u}_1(t)$$

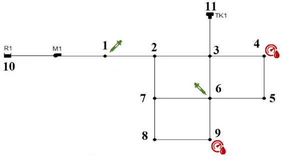
subject to WQM

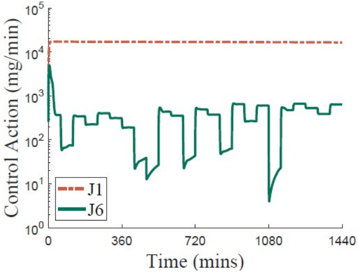
$$\mathbf{x}^{\min} \leq \mathbf{x}(t) \leq \mathbf{x}^{\max},$$

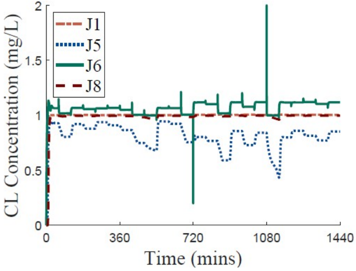
$$\mathbf{u}_1^{\min} \leq \mathbf{u}_1(t) \leq \mathbf{u}_1^{\max},$$

**Minimize Chlorine Injection Cost**

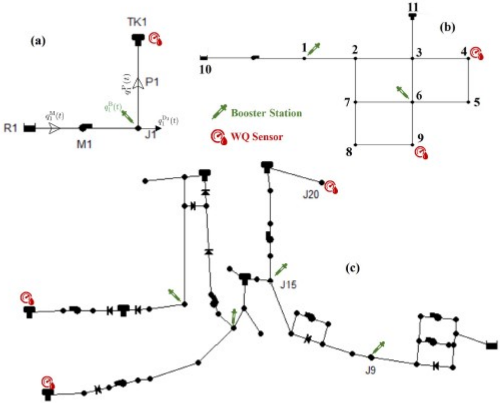
**Water Quality Dynamics Models**



# Booster Stations and Sensors Placement



# Questions and Suggestions?


**Thank You!**