Your Name:

Your Signature:

- **Exam duration:** 2 hours and 1 minute.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything, besides a one-page of formula sheet.
- Solve whatever problems you can solve. The exam is long but (mostly) straightforward. If you've been studying, you'll find it a breather. If you haven't, then I hope the agony is bearable.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**.
- If you need more room, use the back of the pages and indicate that you have done so.
- This exam has 19 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules. Good luck! I hope you do well.

Question Number	Maximum Points	Your Score
1	20	
2	65	
3	25	
4	20	
5	30	
6	40	
Total	200	

1. (20 total points) In the following optimization problem, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ is the optimization variable.

$$\begin{array}{ll} \text{minimize} & e^x + 2e^{-2x} \\ \text{subject to} & \|x - a\|_2 \geq \|x - b\|_2 \end{array}$$

   (a) (20 points) Determine if this problem can be written as a convex optimization problem.

2. (65 total points) Answer the following miscellaneous questions.

   (a) (10 points) Represent the following inequality as a linear matrix inequality, given that A, b, θ are given quantities:

$$||Ax - b|| \leq \theta.$$

(b) (10 points) Let $\mathcal{S}_+^n$ be the set of positive semi-definite matrices of size n. Prove that $\mathcal{S}$ is a convex set.

(c) (10 points) Is the function

$$f(x_1, x_2) = \frac{1}{(x_1 - 2)^2 + (x_2 + 1)^2 + 3}$$

locally convex, concave, or neither in the neighborhood of the point $x^{(0)} = [2\ -1]^T$?

(d) (10 points) Is the quadratic form

$$f = f(x_1, x_2, x_3) = x^T \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} x$$

positive definite, negative semidefinite, or indefinite? Justify your answer carefully.

(e) (15 points) Consider the problem

$$\begin{aligned} \text{minimize} \quad & 2x_1^2 + 3x_2^2 - 4x_1 + 6x_2 + 11 \\ \text{subject to} \quad & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Determine if the problem is convex and show that its solution is the point (1,0).

(f) (10 points) Determine if the function $f(x_1, x_2) = 2x_1^3 - 3x_2^2$ is convex or concave. Then investigate whether there is a subset of $\mathbb{R}^2$ over which the function is convex or concave.

3. (25 total points) The objective of this problem is to show you how LMIs are nothing but nonlinear (but still convex) optimization problems. You are given the following optimization problem:

$$\begin{aligned} \textbf{OP1:}\ \text{minimize}\quad & \textbf{trace}(P) = p_1 + p_2 \\ \text{subject to}\quad & AP + PA^\top + Q = 0 \qquad (1) \\ & P = P^\top \succ 0 \qquad (2) \end{aligned}$$

where $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. For the above problem, assume that $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ is the optimization variable. In other words, you have three variables to solve for, since P is symmetric and positive definite.

(a) (10 points) Define a new variable $x = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top$ and write the first constraint in **OP1** as a linear system of equations, i.e., $\tilde{A}x = b$, where $\tilde{A} \in \mathbb{R}^{4\times 3}$ and $b \in \mathbb{R}^{4\times 1}$ are matrices you should determine.

(b) (5 points) Write the second positive definiteness constraint on P (i.e., $P = P^\top \succ 0$) as a nonlinear set of equations. Remember that a matrix is positive definite if and only if all of its leading principal minors are positive. You should obtain two inequality constraints here.

(c) (10 points) Using the above transformations, write **OP1** as a simple optimization problem with a linear cost function, linear equality constraints, and quadratic inequality constraints. You should get something like this:

$$\textbf{OP2} \equiv \textbf{OP1} \quad \underset{x}{\text{minimize}} \quad c^\top x$$

$$\text{subject to} \quad \tilde{A}x = b \tag{3}$$

$$x_1 > 0 \tag{4}$$

$$x^\top Q x + x^\top \tilde{b} + c > 0 \tag{5}$$

where $c, \tilde{A}, b, Q, \tilde{b}$, and c are constant matrices and vectors that you should have already determined.

4. (20 points) Consider the following problem where $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$, $\lambda_i > 0$ ($i = 1, 2$), are given, and $x \in \mathbb{R}^n$ is the optimization variable.

$$\text{minimize} \;\; \lambda_1 \|A_1 x - b_1\|_2^2 + \lambda_2 \|A_2 x - b_2\|_2^2$$

   (a) (20 points) Determine if the problem is convex and find a closed-form solution. You may assume that matrix $\lambda_1 A_1^T A_1 + \lambda_2 A_2^T A_2$ is invertible.

5. (30 points) Consider the following measurement model

$$y(t) = a^T(t)x + w(t)$$

where $y(t) \in \mathbb{R}$ defines the measurement data, $w(t) \in \mathbb{R}$ depicts measurement noise, vector $a \in \mathbb{R}^n$ is a constant, pre-determined vector, and $x \in \mathbb{R}^n$ is a vector parameter to be estimated. The measurements are defined for $t = 1, 2, \ldots, m$.

   (a) (15 points) Assume that the probability density function of $w(t)$ is given as a Gaussian PDF of

$$p(w) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w-\mu}{\sigma}\right)^2}$$

   where $\mu = 0$ defines the mean and σ the standard deviation. Consider now that our objective is to find a maximum likelihood (ML) estimate for vector x given the m measurements. This problem can be posed as

$$\text{maximize}_{x_{\text{ML}} \in \mathbb{R}^n} \;\; \sum_{t=1}^{m} \log p(y(t); x) = \sum_{t=1}^{m} \log p(y(t) - a^\top(t)x)$$

   Derive the solution to the optimization problem given the above assumptions and show that the ML estimate under Gaussian noise is indeed the solution to a least squares minimization problem.

(b) (15 points) What happens when the noise distribution for $w(t)$ changes to either the Laplacian distribution given by $p(w) = \frac{1}{2a}e^{-|w|/a}$ for a fixed $a > 0$ or the Uniform distribution given by $p(w) = \frac{1}{2a}$ on $[-a, a]$? Derive the maximum likelihood estimates for these two distributions too.

6. (40 points) Consider the following optimization problem

$$\begin{aligned} \min \quad & x^T Q x + c^T x && (6) \\ \text{subject to} \quad & a_i^T x \leq b_i, \quad i = 1, \ldots, m. && (7) \end{aligned}$$

   (a) (25 points) Assume that the scalars b_i, vector c, and matrix Q are all known. Considering now that vectors $a_i \in \mathbb{R}^n$ are all uncertain but yet remain in these ellipsoids centered around a constant $\bar{a}_i$ with orientation/size given by matrices P_i, that is:

$$a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid ||u||_2 \leq 1\}$$

   where matrices $P_i \in \mathbb{R}^{ntimesn}$. Notice that by setting matrix $P_i = 0$, you get a deterministic constraint, i.e., the ith constraint is certain. This form of uncertainty is called ellipsoidal uncertainty. The linear inequality constraint $a_i^T ix \leq b_i$ for all $a_i \in \mathcal{E}_i$, in this problem can be expressed as

$$\sup\{a_i^T x \mid a_i \in \mathcal{E}_i\} \leq b_i, \quad i = 1, \ldots, m.$$

   Given this, show that the above optimization problem can be written as a second order cone program (SOCP). This shows that to *robustify* LPs with some uncertain data, one obtains an SOCP.

(b) (15 points) Rewrite the derived SOCP in the previous problem into *(i)* a semidefinite program (SDP) and *(ii)* a quadratically constrained quadratic program (QCQP), separately. What are the conditions that the problem data has to satisfy so that the QCQP formulation is convex?