VANDERBILT UNIVERSITY
CE 5999
INTRO TO OPTIMIZATION

HOMEWORK # 2

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1. Reading material: Appendices A.4 and A.5.

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2. Are the following sets convex or nonconvex? Justify your answer.

   (a) A sphere with one point on the boundary removed.
   Answer: No. It is clear that sphere with bounday or without bounday are all convex set. In the convex set, exist two points in the bounday, which is very close the point p_0, $x = p_0 + \epsilon$ and $y - \epsilon$, where $\lim \epsilon = 0$, $p_0 = 0.5x + 0.5y$. If the point is removed, the criteria for the convex set is no longer satisfied as we could find two points x and y as the counter case. Hence, it is a nonconvex set.

   (b) A cube with one vertex removed.
   Answer: No. The proof can be similar.

   (c) A cube with one, non-vertex point on the boundary removed.
   Answer: No. The proof can be similar.

   (d) A sphere with one interior point removed.
   Answer: No. The proof can be similar.

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3. Consider two points y and z in $\mathbb{R}^n$. Prove that the set of points that are closer to y than to z is a polyhedron. Write the set in the form $\{x \mid Ax \preceq b\}$.

   Hint: The set of points closer to y than z is written as follows:

$$S = \{x \in \mathbb{R}^n \mid \|x - y\|_2 \leq \|x - z\|_2\}.$$

   Take the square of the previous inequality and express the norm using the inner product.

   Proof: According to the definition, for the point x in set S:

$$\begin{aligned}
\|x - y\|_2 &\leq \|x - z\|_2 && (1)\\
(x - y)^\top (x - y) &\leq (x - z)^\top (x - z) && (2)\\
&= ((x - y) - (z - y))^\top ((x - y) - (z - y)) && (3)\\
&= (x - y)^\top (x - y) + (z - y)^\top (z - y) - 2(x - y)^\top (z - y) && (4)
\end{aligned}$$

   Then, the inequality can be written as follows:

$$(x - y)^\top (z - y) \leq \frac{1}{2}(z - y)^\top (z - y) \quad (5)$$

$$(x - y - \frac{1}{2}(z - y))^\top (z - y) \leq 0 \quad (6)$$

   if $y \preceq z$, $x \preceq \frac{1}{2}(y + z)$;if $z \succeq y$, $x \succeq \frac{1}{2}(y + z)$

   Hence, $(z - y)^\top x \leq \frac{1}{2}(z - y)^\top (y + z) = \frac{1}{2}(||z||_2 - ||y||_2)$.

   Here, for the definition of Polyhedron, $A = (z - y)^\top$, $b = \frac{1}{2}(||z||_2 - ||y||_2)$.

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4. Prove that the set of $n \times n$ symmetric matrices is a subspace.

   Hint: Use the definition of subspace.

   Proof: Matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices $\mathbf{S}^n$. $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the following property:

$$\boldsymbol{A}^\top = \boldsymbol{A}, \boldsymbol{B}^\top = \boldsymbol{B} \tag{7}$$

   For any matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ and any k,

$$(\boldsymbol{A}+\boldsymbol{B})^\top = \boldsymbol{A}^\top + \boldsymbol{B}^\top \tag{8}$$

$$(k\boldsymbol{A})^\top = k\boldsymbol{A}^\top \tag{9}$$

   for any λ and any $\mu \in \mathbb{R}$,

$$(\lambda \boldsymbol{A} + \mu \boldsymbol{B})^\top = (\lambda \boldsymbol{A})^\top + (\mu \boldsymbol{B})^\top \tag{10}$$

$$= \lambda \boldsymbol{A}^\top + \mu \boldsymbol{B}^\top = \lambda \boldsymbol{A} + \mu \boldsymbol{B} \tag{11}$$

   Hence, the set of $n \times n$ symmetric matrices is a subspace.

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5. *Voronoi* description of halfspace. Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.
   Proof: I think the proof is similar to Problem 3. Just replace x, y with a, b.
   According to the definition, for the point x in set X:

$$\|x-a\|_2 \leq \|x-b\|_2 \tag{12}$$

$$(x-a)^\top(x-a) \leq (x-b)^\top(x-b) \tag{13}$$

$$= ((x-a)-(b-a))^\top((x-a)-(b-a)) \tag{14}$$

$$= (x-a)^\top(x-a) + (b-a)^\top(b-a) - 2(x-a)^\top(b-a) \tag{15}$$

   Then, the inequality can be written as follows:

$$(x-a)^\top(b-a) \leq \frac{1}{2}(b-a)^\top(b-a) \tag{16}$$

$$(x-a-\frac{1}{2}(b-a))^\top(b-a) \leq 0 \tag{17}$$

   if $a \preceq b$, $x \preceq \frac{1}{2}(a+b)$;if $b \succeq a$, $x \succeq \frac{1}{2}(a+b)$

   Hence, $(b-a)^\top x \leq \frac{1}{2}(b-a)^\top(a+b) = \frac{1}{2}(||b||_2 - ||a||_2)$.

   Here, for the definition of Polyhedron, $c = (b-a)$, $d = \frac{1}{2}(||b||_2 - ||a||_2)$.

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6. Which of the following sets are convex?

   (a) A slab, i.e., a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.
   Answer:A slab is an intersection of two halfspaces. Hence, it is a convex set.

   (b) A rectangle, i.e., a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$. A rectangle is sometimes called a hyperrectangle when $n > 2$.
   Answer: Convex. A rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

   (c) A wedge, i.e., $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$.
   Answer: Convex. A wedge is an intersection of two halfspaces, so it is convex set.

(d) The set of points closer to a given point than a given set, i.e.,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.
Answer: Convex. This is an intersection of halfspaces.

(e) The set of points closer to one set than another, i.e.,

$$\{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\operatorname{dist}(x, S) = \inf \{\|x - z\|_2 \mid z \in S\}.$$

Answer: Nonconvex. Find a set $S = \{0, 2\} \in \mathbb{R}$, $T = \{1\}$, then: $\{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\} = \{x \in \mathbb{R} \mid x \leq 0.5 \text{ or } x \geq 1.5\}$. Obviously, it is nonconvex.

(f) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.
Answer: Convex.This is an intersection of two convex sets.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$
Answer: Convex (ball).

$$\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\} \tag{18}$$

$$= \left\{x \mid \|x - a\|_2^2 \leq \theta^2 \|x - b\|_2^2\right\} \tag{19}$$

$$= \left\{x \mid \left(1 - \theta^2\right) x^T x - 2\left(a - \theta^2 b\right)^T x + \left(a^T a - \theta^2 b^T b\right) \leq 0\right\} \tag{20}$$

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7. Positive semidefinite cone for $n = 1, 2, 3$. Give an explicit description of the positive semidefinite cone $\mathbf{S}_+^n$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element of $\mathbf{S}^n$, for $n = 1, 2, 3$, use the notation

   Hint: Use the following result we covered in class: A symmetric $n \times n$ matrix is positive semidefinite if and only if all its principal minors are nonnegative. Check out the slides on what principal minors are.

$$x_1, \quad \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}$$

Solution:

For $n = 1$ the condition is $x_1 \geq 0$.

For $n = 2$ the condition is

$$x_1 \geq 0, \quad x_3 \geq 0, \quad x_1 x_3 - x_2^2 \geq 0.$$

For $n = 3$ the condition is

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_1 x_4 - x_2^2 \geq 0, \quad x_4 x_6 - x_5^2 \geq 0, \quad x_1 x_6 - x_3^2 \geq 0$$

and

$$x_1 x_4 x_6 + 2 x_2 x_3 x_5 - x_1 x_5^2 - x_6 x_2^2 - x_4 x_3^2 \geq 0,$$
