HOMEWORK # 4

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**Due date of the homework: March 5th, midnight. Most of these problems have very short answers, so do not be surprised. They just require some thinking.**

1. Given $M$ points in $\mathbb{R}^n$, that is, $y^{(1)}, \dots, y^{(M)} \in \mathbb{R}^n$, find a point with the property that the sum of squared Euclidean distances from $y^{(i)}$ is minimized. The problem is formulated as follows:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{M} \|x - y^{(i)}\|_2^2$$

Note that here the distances are squared. You should be able to find the solution in closed form.

Let $f(x, y^{(1)}, \dots, y^{(M)}) = \sum_{i=1}^{M} \|x - y^{(i)}\|_2^2$. Then we have:

$$\begin{aligned} f(x, y^{(1)}, \dots, y^{(M)}) &= \sum_{i=1}^{M} (x - y^{(i)})^T (x - y^{(i)}) \\ &= \sum_{i=1}^{M} (x^T x - 2x^T y^{(i)} + y^{(i)T} y^{(i)}) \end{aligned}$$

Let $\frac{\partial f(x, y^{(1)}, \dots, y^{(M)})}{\partial x} = 0$, then we can get:

$$\sum_{i=1}^{M} (2x - 2y^{(i)}) = 0$$

$$\Rightarrow x = \frac{\sum_{i=1}^{M} y^{(i)}}{M}$$

2. Boyd-Vandenberghe 4.1.

(a): the optimal point: $(\frac{2}{5}, \frac{1}{5})$, the optimal value: $\frac{3}{5}$.
(b): there is no optimal set and the optimal value is unbounded.
(c): optimal set: $\{(x_1, x_2) | x_1 = 0, x_2 \geq 1\}$, optimal value: 0.
(d): optimal point: $(\frac{1}{3}, \frac{1}{3})$, optimal value: $\frac{1}{3}$.
(e): optimal point: $(\frac{1}{2}, \frac{1}{6})$, optimal value: $\frac{1}{2}$

3. Boyd-Vandenberghe 4.3.

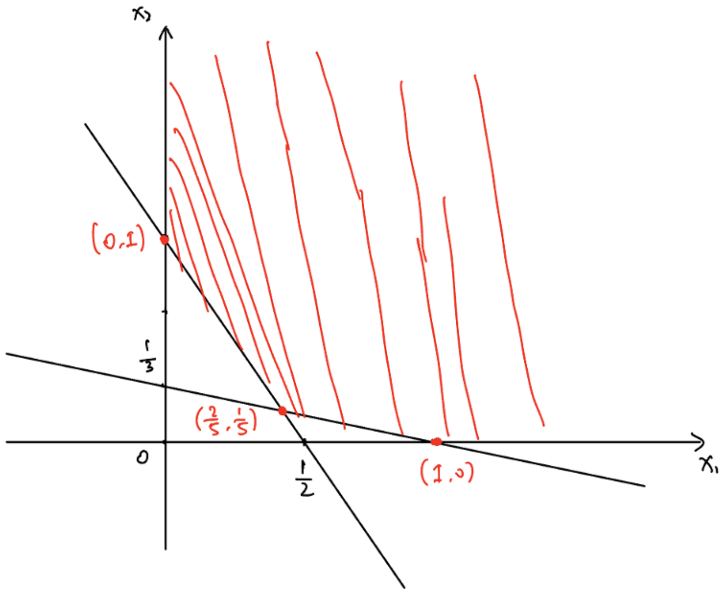


Figure 1: the sketch of the feasible set

We need to prove that $\nabla f_0(x^*)^T(x - x^*) \geq 0$ for all $x \in C$:

$$\nabla f_0(x^*) = Px^* + q = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

then we have:

$$\nabla f_0(x^*)^T(x - x^*) = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - \frac{1}{2} \\ x_3 + 1 \end{bmatrix}$$
$$= -(x_1 - 1) + 2(x_3 + 1) \geq 0 \text{for} x_1, x_2, x_3 \in [-1, 1]$$

Therefore, the given point is optimal for this optimization problem.

*Hint:* Apply the optimality condition we learned in class for any optimal point.

4. Boyd-Vandenberghe 4.9.

We change variables: $y = Ax$, then we have $x = A^{-1}y$:

$$\begin{aligned} \text{minimize} \quad & c^T A^{-1} y \\ s.t. \quad & y \preceq b \end{aligned}$$

The objective function is a linear combination of the elements of $y$ with coefficient as elements of $c^T A^{-1}$. If $c^T A^{-1} \preceq 0$, then all coefficients are non-positive and we can get a minimal value when $y = b$. Otherwise the objective value could be negative infinity since it is unbounded.

*Hint:* Apply a change of variables $y = Ax$ and then observe the resulting optimization objective function and constraints.

5. Boyd-Vandenberghe 4.11.

(a) minimize $||Ax - b||_\infty$ is equivalent to minimize $|a_i^T x_i - b_i|$ for all $i$ where $a_i^T$ is the $i$th row in matrix $A$ and $b_i$ is the $i$th element of vector $b$. Then the equivalent LP is:

$$\begin{aligned} \textit{minimize} \quad & t \\ s.t. \quad & |a_i^T x_i - b_i| \leq t \quad \textit{for all } i \end{aligned}$$

(b) the original problem is equivalent to: minimize $\sum_{i=1}^m |a_i^T x_i - b_i|$ for all $i$ where $a_i^T$ is the $i$th row in matrix $A$ and $b_i$ is the $i$th element of vector $b$. Then the equivalent LP is:

$$\begin{aligned} \textit{minimize} \quad & mt \\ s.t. \quad & |a_i^T x_i - b_i| \leq t \quad \textit{for all } i \end{aligned}$$

(c) this is similar with (b) but with additional constraints:

$$\begin{aligned} \textit{minimize} \quad & mt \\ s.t. \quad & |a_i^T x_i - b_i| \leq t \quad \textit{for all } i \\ & x_i \leq t \quad \textit{for all } i \end{aligned}$$

(d) this is equivalent to:

$$\begin{aligned} \textit{minimize} \quad & mt \\ s.t. \quad & |x_i| \leq t \quad \textit{for all } i \\ & |a_i^T x_i - b_i| \leq 1 \quad \textit{for all } i \end{aligned}$$

(e) this is equivalent to:

$$\begin{aligned} \textit{minimize} \quad & mt + s \\ s.t. \quad & |x_i| \leq s \quad \textit{for all } i \\ & |a_i^T x_i - b_i| \leq t \quad \textit{for all } i \end{aligned}$$

*Hint:* Explain the equivalence of the given problems and the LPs you derive.

6. Boyd-Vandenberghe 4.12. Read the problem description carefully.

$$\begin{aligned} \textit{minimize} \quad & C = \sum_{i,j=1}^{n} c_{ij} x_{ij} \\ s.t. \quad & \sum_{k=1}^{n} x_{ki} + b_i - \sum_{k=1}^{n} x_{ik} = 0 \ \textit{for all } i \\ & \mathbf{1}^T b = 0 \\ & l_{ij} \leq x_{ij} \leq u_{ij} \ \textit{for all } i, j \end{aligned}$$

7. Boyd-Vandenberghe 4.13.

$$\begin{aligned} \textit{minimize} \quad & c^T x \\ s.t. \quad & \bar{A}x + Vx \preceq b \\ & \bar{A}x - Vx \preceq b \end{aligned}$$

*Hint:* The problem is simpler than you might think. Use absolute values. ;)

8. Boyd-Vandenberghe 4.15.

(a) It is obvious that the relaxation problem could generate a smaller objective value than the original problem since the feasible set is larger. The Boolean LP will also be infeasible if LP relaxation is infeasible. (b) If LP relaxation has a solution in the original feasible set then this solution is also optimal for the original problem.

9. Boyd-Vandenberghe 4.16.

$$\begin{aligned} \textit{minimize} \quad & F = \sum_{t=0}^{N-1} ((1 - x(t))|u(t)| + x(t)(2|u(t)| - 1)) \\ \textit{s.t.} \quad & \sum_{t=0}^{N-1} A^{N-1-t} b u(t) = x_{des} \\ & x(t) \in \{0, 1\} \end{aligned}$$

where $x(t) = 0$ if $|u(t)| \le 1$ otherwise $x(t) = 1$.

10. Boyd-Vandenberghe 4.23.

This is equivalent to minimize $||Ax - b||_4^4 = \sum_{i=1}^m (a_i^T x - b_i)^4$, which is in turn equivalent to the following QCQP:

$$\begin{aligned} \textit{minimize} \quad & \sum_{i=1}^{m} z_i^2 \\ \textit{s.t.} \quad & y_i = a_i^T x - b_i \\ & y_i^2 \le z_i \end{aligned}$$

11. Boyd-Vandenberghe 4.26-a). Verify the claim first then solve part a) of the problem.

Verification:

$$
\begin{aligned}
\left\|\begin{bmatrix} 2x \\ y-z \end{bmatrix}\right\|_2 &\le y+z \\
\Leftrightarrow \sqrt{4x_1^2 + 4x_2^2 + \cdots + 4x_n^2 + (y-z)^2} &\le y+z \\
\Leftrightarrow 4x_1^2 + 4x_2^2 + \cdots + 4x_n^2 + (y-z)^2 &\le (y+z)^2 \\
\Leftrightarrow x_1^2 + x_2^2 + \cdots + x_n^2 &\le yz \\
\Leftrightarrow x^T x &\le yz
\end{aligned}
$$

(a) the original problem is equivalent to:

$$
\begin{aligned}
minimize \quad & \sum_{i=1}^{m} \frac{1}{a_i^T x - b_i} \\
s.t. \quad & a_i^T x - b_i \ge 0 \ for\ all\ i
\end{aligned}
$$

which is equivalent to:

$$
\begin{aligned}
minimize \quad & \sum_{i=1}^{m} t_i \\
s.t. \quad & t_i(a_i^T x - b_i) \ge 1 \\
& a_i^T x - b_i \ge 0 \ for\ all\ i
\end{aligned}
$$

which can be transferred to:

$$
\begin{aligned}
minimize \quad & \sum_{i=1}^{m} t_i \\
s.t. \quad & \left\|\begin{bmatrix} 2 \\ t_i - a_i^T x + b_i \end{bmatrix}\right\|_2 \le t_i + a_i^T x - b_i \\
& a_i^T x - b_i \ge 0 \ for\ all\ i
\end{aligned}
$$

12. Boyd-Vandenberghe 4.33.

(a): this is equivalent to:

$$\begin{aligned} minimize \quad & t \\ s.t. \quad & p(x) \leq t \\ & q(x) \leq t \end{aligned}$$

which is a geometric program and we can transfer it into a convex form by changing of variables: $y_i = \log x_i$.
(b): this is equivalent to:

$$\begin{aligned} minimize \quad & exp(t_1) + exp(t_2) \\ s.t. \quad & p(x) \leq t_1 \\ & q(x) \leq t_2 \end{aligned}$$

we can then transfer it into a convex form by changing of variables: $y_i = \log x_i$.
(c): this is equivalent to:

$$\begin{aligned} minimize \quad & t \\ s.t. \quad & p(x) \leq t(r(x) - q(x)) \\ & r(x) > q(x) \end{aligned}$$

we can then transfer it into a convex form by changing of variables: $y_i = \log x_i$.