Due date of the homework: February 3rd, 2023, midnight.

1. Compute the gradient and the Hessian of this function

$$f(x) = f(x_1, x_2, x_3) = 3x_1^2 x_3 + 2x_3 - 4x_3 x_2^2$$

   at this operating point $x^{(0)} = [3 \ -1 \ 2]^\top$.

2. Compute the gradient and the Hessian of this function

$$f(x) = f(x_1, x_2) = 3x_1^2 cos(e^{-x_2}) + 2x_2^2 + 7x_1 - 8x_2 + x_1^2 + 4$$

   at this operating point $x^{(0)} = [0 \ -1]^\top$.

3. For Problems 1 and 2, do the following:
   - Compute the first order and second (quadratic) order Taylor series approximation around the given operating points.
   - Using Matlab, plot the functions and their corresponding approximations.
   - Evaluate the definiteness of the quadratic approximations (i.e., are the resulting quadratic approximation positive definite, positive semidefinite, negative definite, etc...).

4. Compute the eigenvalues, eigenvectors, determinant, and rank of these two matrices

$$A = \begin{bmatrix} 1 & \pi \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

   The computations should be by hand but please verify your answers via Matlab. We did not cover that in Module 0, so feel free to use online content to guide you through how matrix properties are manually computed.

5. Verify that the one-norm given by

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

   is indeed a norm on $\mathbb{R}^n$ via proving that it satisfies the three vector norm properties.

6. Prove that for all vectors $x \in \mathbb{R}^n$, we have

$$||x||_1 \geq ||x||_2.$$

7. Compute $||A||_F$, $||A||_2$, $||A||_1$ and $||A||_\infty$ of this matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}.$$

8. Prove that this new norm for any square matrix A with dimension n defined as

$$||A||_{**} = \max|a_{ij}|, \quad \text{for all} \ \ 1 \leq i, j \leq n$$

   is not a legit norm. You can find a counter example that contradicts one of the basic matrix norm properties.

9. Prove that the Frobenius norm of any matrix A is indeed a legitimate matrix norm by showing that it satisfies the basic matrix norm properties.

10. Find the *full* and *thin* singular value decompositions for these two matrices by hand and verify them via Matlab

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

    Slide 44/50 in Module 0 explains how to compute these decompositions.

11. Compute the 2- and nuclear-norms of matrices A and B given in the previous problems (by utilizing the results of your SVD).