HOMEWORK # 3

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February 11, 2023

Due date of the homework: February 21st, midnight.

1. Determine whether or not the following functions are convex in their given domains:
   (a) $f(x) = x_1 x_2$ in the domain $x \in \mathbb{R}^2$
   (b) $f(x) = e^{x_1+x_2}$ in the domain $x \in \mathbb{R}^2$
   (c) $f(x) = x_1^4 + x_2^4 - x_1^2 x_2^2$ in the domain $x \in \mathbb{R}^2$
   (d) $f(x) = x_1^3 + x_2^3$ in the domain $x \in \mathbb{R}^2_{++}$
   (e) $f(x) = \tan x$ for the domain $x \in (0, 1)$ (boundary points not included)
   (f) $f(x) = \frac{1}{x_1 x_2}$ in the domain $x \in \mathbb{R}^2_{++}$.
   (g) $f(x) = \frac{x_1}{x_2}$ in the domain $x \in \mathbb{R}^2_{++}$.
   (h) $f(x) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \le \alpha \le 1$ and in the domain $x \in \mathbb{R}^2_{++}$.

2. Via the definition of a convex function (Jensen's inequality

$$f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2)$$

   for $\theta \in [0, 1]$), prove that the quadratic function

$$f(x) = \frac{1}{2} x^\top Q x - x^\top b + c$$

   for all $x \in \mathbb{R}^n$ is convex if the symmetric matrix Q is positive semidefinite.

3. Is the product of two convex functions convex? If yes, prove it; if not, give a counterexample.

4. A given function $f : \mathbb{R}^n \to R$ is assumed to be continuously differentiable. Also, assume that $f(x)$ **is concave on a convex set** $\mathcal{X}$. Given the aforementioned properties of $f(x)$, prove that for all $x_1, x_2 \in \mathcal{X}$, $f(x)$ satisfies this property:

$$f(x_2) \le f(x_1) + Df(x_1)(x_2 - x_1).$$

   Hint: *Back to basics—what is the basic definition of a derivative?*

5. Show that the set Ω given by $\Omega = \{y \in \mathbb{R}^2; \|y\|^2 \le 4\}$ is convex, where $\|y\|^2 = y^\top y$.

6. Given a multivariable function $f(x)$, many optimization solvers use the following algorithm to solve $\min_x f(x)$:
   (a) Choose an initial guess, $x^{(0)}$
   (b) Choose an initial real, symmetric positive definite matrix $H^{(0)}$
   (c) Compute $d^{(k)} = -H^{(k)} \nabla_x f(x^{(k)})$
   (d) Find $\beta^{(k)} = \arg\min_\beta f(x^{(k)} + \beta^{(k)} d^{(k)}),\ \beta \ge 0$
   (e) Compute $x^{(k+1)} = x^{(k)} + \beta^{(k)} d^{(k)}$

   For this problem, we assume that the given function is a typical quadratic function ($x \in \mathbb{R}^n$), as follows:

$$f(x) = \frac{1}{2} x^\top Q x - x^\top b + c, \quad Q = Q^\top \succ 0.$$

   Answer the following questions:

(a) Find $f(x^{(k)} + \beta^{(k)} d^{(k)})$ for the given quadratic function.

(b) Obtain $\nabla_x f(x^{(k)})$ for $f(x)$.

(c) Using the chain rule, and given that $\beta^{(k)} = \arg\min_\beta f(x^{(k)} + \beta^{(k)} d^{(k)})$, find a closed form solution for $\beta^{(k)}$ in terms of the given matrices ($H^{(k)}, \nabla f(x^{(k)}), d^{(k)}, Q$).

(d) Since it is required that $\beta^{(k)} \geq 0$, provide a sufficient condition related to $H^{(k)}$ that guarantees the aforementioned condition on $\beta^{(k)}$.

7. For the following function, find the set of values for β such that the function is convex.

$$f(x, y, z) = x^2 + y^2 + 5z^2 - 2xz + 2\beta xy + 4yz$$

8. Prove that the following set given by

$$\Psi = \{x : x^\top P x \leq 1\}$$

is in fact a convex one given that the matrix P is a symmetric positive definite matrix.

9. Now given the previous problem, prove that

$$g(x) = x_1^2 + 2x_1x_2 + 3x_2^2 - 5x_1 + 6x_2 + 10$$

is a convex function by writing $g(x) = x^\top P x + b^\top x + c$ and then using the argument that the sum of convex function is a convex function.

10. Boyd-Vandenberghe 3.15.

11. Boyd-Vandenberghe 3.18.