HOMEWORK # 5

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Due date of the homework: April 13th, midnight.

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1. Solve Lab 1 (available here: https://ecal.berkeley.edu/files/ce191/Lab1.pdf) from my colleague's optimization class at UC Berkeley. I was gonna copy the problem herein, but I see no point in doing so. The problem seems too long, but it isn't once you formulate the linear program. Most of the other exercises are basic analysis and changes in parameters for the original problem formulation.

2. You are given the set of all 2×2 matrices with diagonal elements $(1, 2)$ which we can write as

$$\mathcal{R} = \left\{ \begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix}, \; x, y \in \mathbb{R} \right\}.$$

   (a) Is the set $\mathcal{R}$ convex?
   (b) Are rank constraints on matrices convex or nonconvex?
   (c) Defining a subset of $\mathcal{R}$ as $\mathcal{R}_1 \subset \mathcal{R}$ to be the set of rank-one matrices. Derive conditions on x and y that allow for all matrices $\mathcal{R}_1 \subset \mathcal{R}$ to be rank-one.
   (d) Is the set $\mathcal{R}_1$ convex?
   (e) Write the following optimization problem

$$\text{minimize} \quad ||A||_F^2 \quad \text{subject to} \quad A \in \mathcal{R}_1$$

   as an explicit optimization problem, where $||A||_F$ is the Frobenius norm. Is this problem convex?

3. (***Reading Section 4.6.2 from the textbook can help in solving this problem***.) Consider the following primal SDP:

$$p^* = \text{minimize} \;\; x \quad \text{subject to} \;\; \begin{bmatrix} 1 & x+y \\ x+y & y \end{bmatrix} \succeq 0, \;\; x, y \in \mathbb{R}$$

   (a) Show that this SDP is strictly feasible.
   (b) Find the dual of the primal SDP. This will be an SDP in standard form.
   (c) Show that the dual SDP is infeasible.
   (d) Since the primal SDP is strictly feasible we know that strong duality holds. However, the dual SDP is infeasible. What does this say about the primal value p^*? Can you directly justify this?

4. Consider the well-known constrained, least-squares problem:

$$\text{minimize} \quad ||Ax - b||_2^2 \quad \text{subject to} \quad Gx = h$$

   where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, G \in \mathbb{R}^{p \times n}$ and $h \in \mathbb{R}^p$. Assume that $\text{rank}(A) = n$ and $\text{rank}(G) = p$.

   (a) Using the KKT conditions, determine the optimal solution of this optimization problem.

5. Let $p^{(1)}, \ldots, p^{(r)}$ and $q^{(1)}, \ldots, q^{(s)}$ be different vectors in $\mathbb{R}^d$, where $r, s \geq 1$. Let $\mathcal{P}$ denote the polytope defined as the convex hull of the $p^{(i)}$'s, and $\mathcal{Q}$ the polytope defined as the convex hull of of the $p^{(i)}$'s (check the textbook definition on what constitutes a convex hull and polytope).

   Define a new matrix $C \in \mathbb{R}^{d \times n}$ whose i-th column is $p^{(i)}$ for all $1 \leq i \leq r$ and whose $(r+j)$-th column is $-q^{(j)}$ for all $1 \leq j \leq s$.

(a) Given the above, formulate an optimization problem of computing the minimum $\mathcal{L}_2$ squared distance between points in $\mathcal{P}$ and $\mathcal{Q}$. The problem should be written as a QP with an objective function $||Cx||_2^2$ where $x \in \mathbb{R}^n$.

(b) Let $y = Cx$. Demonstrate that the previous QP formulated in the previous part of this problem can be written as a QP with the objective function $||y||_2^2$, with two optimization variables x and y of appropriate dimensions.

(c) Derive the dual of part b) and showcase that the dual objective function maximization can be written as

$$d^* = \underset{\lambda}{\text{maximize}} \left(-\frac{1}{4}\lambda^\top \lambda + \underset{i=1,\dots,r}{\text{minimize}} \ \ \lambda^\top p^{(i)} - \underset{j=1,\dots,s}{\text{maximize}} \ \ \lambda^\top q^{(j)} \right).$$

6. You are given the following optimization problem:

$$\text{minimize } f(x) = x_1^2 + x_2^2 \ \ \text{subject to} \ \ (x_1-1)^2 + (x_2-1)^2 \leq 2 \ , \ \ (x_1-1)^2 + (x_2+1)^2 \leq 2.$$

(a) Draw the feasible region of this problem on Matlab. Furthermore, on top of the figure sketch he the level sets of $f(x)$.

(b) Is this problem convex? Verify that strong duality holds for this problem via Slater's condition.

(c) Solve the problem via deriving the KKT conditions. There are a few cases to consider for the multipliers. List them all. Trivial solutions without justifications are not accepted.

7. For this given SDP

$$\textbf{(LMI-1)} \quad \min \quad \text{trace}(P) \quad \text{subject to} \quad A^\top P + PA - C^\top Y^\top - YC \prec 0, \quad P = P^\top \succ 0,$$

(where matrices P and Y are the optimization variables and matrices A and C are constant) derive the SDP form in the LMI form. This form can be written as $x \in \mathbb{R}^n$

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} & \sum_{i=1}^{n} x_i F_i + G \preceq 0 \\ & Dx = b \end{aligned}$$

where $F_1, \dots, F_n, G \in \mathbb{S}^m$ (i.e., $F_1, \dots, F_n, G$ are **symmetric** constant matrices). Essentially, I want you to derive these constant matrices for **(LMI-1)** given above.

Hint: Module 5 (the SDP slides, specifically the part titled *LMIs with matrix variable*) can help solving this problem.

8. Solve the previous problem but for a different SDP given as follows

$$\textbf{(LMI-2)} \quad \min \quad ||P||_1 \quad \text{subject to} \quad \begin{bmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & D^\top D - I \end{bmatrix} \prec 0, \quad P = P^\top \succ 0,$$

where matrices B, C, and D are all constant. Matrix P is the only matrix variable. Note that for this problem you need to utilize the epigraph trick to model the L_1 norm objective function into a linear objective function with some constraints.

9. Show that the following bilinear matrix inequalities (or in general any BMI) are nonconvex.

$$\textbf{(BMI-1)} \quad \text{find} \quad P, \alpha \quad \text{subject to} \quad A^\top P + PA - \alpha P \prec 0, \quad P = P^\top \succ 0, \alpha > 0$$

where α is a scalar.

$$\textbf{(BMI-2)} \quad \text{find} \quad P, Y \quad \text{subject to} \quad A^\top P + PA - Y^\top P - PY \prec 0, \quad P = P^\top \succ 0$$

where P and Y are matrix variables.