HOMEWORK # 6

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Due date of the homework: May 4th, midnight. FIRM DEADLINE. Final grades are due from faculty on May 6th.

1. You are given the following optimization problem

$$\text{minimize} \quad f(x) = x_1^2 + x_2^2 - 8x_1 + 2x_2 + 10.$$

   (a) Analytically, and via the computation of the gradient and Hessian of $f(x)$, compute x^* the optimal solution of the unconstrained optimization problem.

   (b) The iterates of the gradient descent and Newton's method can be written as

$$\text{Gradient-Descent}: \quad x_{k+1} = x_k - t_k \nabla f(x_k) \quad \text{Newton}: \quad x_{k+1} = x_k - t_k H_k^{-1} \nabla f(x_k)$$

   where t_k is the step size and H_k is the Hessian matrix at iteration index k.
   Starting from an initial guess $x_0 = [8\ 3]^\top$, code using Matlab/Python 100 iterations of the two algorithms. Use a constant step-size $t_k = 0.1, 0.2, \ldots, 1$ and apply the algorithm accordingly. Then plot the direction/path towards the optimal solution for each of these time-steps. You might wanna use a logarithmic plot to show the convergence as we did in the module.

   (c) What do you conclude here?

2. You are given the following optimization problem

$$\text{minimize} \quad f(x) = \cosh(0.05x_1^2 + x_2^2).$$

   (a) Can you analytically, and via the computation of the gradient and Hessian of $f(x)$, compute x^* the optimal solution of the above unconstrained optimization problem?

   (b) The iterates of the gradient descent and Newton's method can be written as

$$\text{Gradient-Descent}: \quad x_{k+1} = x_k - t_k \nabla f(x_k) \quad \text{Newton}: \quad x_{k+1} = x_k - t_k H_k^{-1} \nabla f(x_k)$$

   where t_k is the step size and H_k is the Hessian matrix at iteration index k.
   Starting from an initial guess $x_0 = [-2\ 0.9]^\top$, code using Matlab/Python thousands of iterations of the two algorithms until convergence is achieved. Use a constant step-size $t_k = 0.1, 0.2, \ldots, 1$ and apply the algorithm accordingly. Then plot the direction/path towards the optimal solution for each of these time-steps. You might wanna use a logarithmic plot to show the convergence as we did in the module.

3. You are given the following function

$$f(x) = 1.5(x_1^2 + x_2^2) + (1+a)x_1x_2 - (x_1 + x_2) + b.$$

   where a and b are unknown parameters.

   (a) Find the largest set of values for a and b such that a global minimizer exists for this function and compute x^* in terms of these two parameters.

   (b) Apply

$$\text{Gradient-Descent}: \quad x_{k+1} = x_k - 0.4\nabla f(x_k)$$

   to the above optimization problem to find the largest set of values for a and b for which the above gradient-descent algorithm converge to the global minimizer for any initialization point.

4. Go through the Model Predictive Control module I posted on Brightspace. Then, if you want to make your life easier for the next problems, read the first two chapters of Liuping Wang's *Model Predictive Control System Design and Implementation Using MATLAB.* The book is available as a PDF online.

5. You are given this dynamical system that represents a discrete-time direct current (DC) motor:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0.9048 & 0 \\ 0.0952 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0.0952 \\ 0.0048 \end{bmatrix}}_{B} u(k), \quad y(k) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where the two states are $x_1(k)$ and $x_2(k)$; the control input is $u(k)$; the output is $y(k) = x_2(k)$.

(a) Starting from any zero initial conditions, create a model predictive control that ensures that the control input and its deviation are constrained as follows

$$0 \leq u(k) \leq 0.6, \quad -0.2 \leq \Delta u(k) = u(k) - u(k-1) \leq 0.2.$$

Use a prediction horizon equal to a control horizon $N_p = N_c = 30$. The objective function of your MPC minimizes only your control input through the R-matrix which is set to $R = I$.

I have to point out here that are so many ways to solve this problem. The first way is to dump everything into CVX/Yalmip as I mentioned in class. The second way that I want you to pursue is via the approach in the slides/textbook mentioned above.

(b) We now want the states to be both constrained within the values 1 and -1. Re-solve the optimization MPC and include a tracking signal for $x_2(k) = y(k)$ to be set as $r(k) = 0.5$. Initially, use a Q-matrix equal to identity, then tune the weight of the Q and R matrices to achieve better results.

Make sure to include your Matlab/Python codes, not as screenshots but rather in the latex environment (like verbatim for example).

6. Solve Problems 16.6, 16.7, and 16.8 below. These problems are supplemental exercises included with Boyd's textbook (but not included within the textbook). Include the codes in your solutions.

16.6 *Planning an autonomous lane change.* A vehicle is traveling down a highway with two lanes, separated by L meters. At time t, its position is $p(t) = (x(t), y(t)) \in \mathbf{R}^2_+$. We require that $y(t) \in [0, L]$, for all t. When $y(t) = 0$, it means the vehicle is in lane 1, when $y(t) = L$, it means the vehicle is in lane 2, and when $0 < y(t) < L$, it means the vehicle is passing between lanes. (Notice that since a lane on a highway has traffic moving in a single direction, we require that $x(t)$ is nondecreasing in t.)

For simplicity, we discretize the problem. We will consider the position of the vehicle every second, so $p_t = (x_t, y_t)$, $t = 0, 1, \ldots, T$, denotes the vehicles position from 0 to T seconds (in particular, this means $p_t = p(t)$). Initially ($t = 0$), the vehicle lies in lane 1, and we assume $x_0 = 0$. Between t and $t + 1$ seconds, we assume the vehicle travels at constant speed, measured in meters per second (m/s). The speed from time t to $t + 1$ is simply $\|p_{t+1} - p_t\|_2$. We require that these speeds never exceed $S^{\max}$ (for example, the speed limit plus, say, 4 or 5 m/s).

The goal of this problem is to plan a lane change. In particular, after time T^{start}, the vehicle may initiate a lane change from lane 1, and by time T^{end}, the vehicle should have fully entered lane 2.

The vehicle should always travel at a speed of at most $S^{\max}$, measured in meters per second. Additionally, when the vehicle is not allowed to lane change (before T^{start} and after T^{end}), the vehicle must be driving with at least a given minimum speed, $S^{\min}$, which is also given. (You may assume that T^{start} and T^{end} are integers.)

Your goal is to determine the smoothest possible lane change, subject to the constraints described above. By smooth, we simply mean that you should minimize the total acceleration of the vehicle, which can be approximated by

$$\sum_{t=1}^{T-1} \|(p_{t+1} - p_t) - (p_t - p_{t-1})\|_2^2.$$

(a) Explain how to plan this autonomous lane change using convex or quasiconvex optimization, given T, T^{start}, T^{end}, $S^{\min}$, $S^{\max}$, and L. If you introduce new variables or make any transformations you must justify them.

(b) Carry out this method on the data below,

$$T = 30, \quad T^{\text{start}} = 15, \quad T^{\text{end}} = 20, \quad S^{\min} = 25, \quad S^{\max} = 35, \quad L = 3.7.$$

Produce a plot the speed of the vehicle against time, as well as the position of the vehicle in $\mathbf{R}^2$ for the plan you produce.

Remark. In fact, many highways have lanes separated by 3.7 meters. Additionally, on average, a lane change for a vehicle on a standard US freeway takes 5 to 6 seconds, and the speed limits we impose here correspond to a vehicle driving between 55 mph, and 75 mph, which aren't unreasonable for a standard US highway.

16.7 *Optimal racing of an energy-limited vehicle.* We have an energy-limited vehicle, such as a solar car, moving along a fixed straight track. We'd like to design a control system to move the vehicle from the starting point to the finishing point using minimum energy in the time interval $[0, T]$. (There are other related natural formulations of this problem, such as traversing the track in the minimum time subject to a maximum energy usage. We will not consider these here, but the same techniques are applicable.)

At time t the car has position $x(t) \in \mathbf{R}$, velocity $v(t) \in \mathbf{R}$ and acceleration $a(t) \in \mathbf{R}$. The car starts with $x(0) = 0$ and $v(0) = 0$ and must finish with $x(T) \geq x^{\text{final}}$.

At time t the kinetic energy of the vehicle is $k(t) = \frac{1}{2}mv(t)^2$, where m is the mass. Let the energy delivered from the battery to the drivetrain be $p(t)$, which is nonnegative (there is no regenerative braking.) Then

$$\dot{k}(t) = p(t) - p^{\text{brake}}(t) - p^{\text{loss}}(t)$$

where $p^{\text{brake}}(t) \geq 0$ is an input that the control system (*i.e.*, your optimization) chooses, and losses due to drag are modeled via

$$p^{\text{loss}}(t) = c^{\text{loss}} v(t)^3$$

Here c^{loss} is a positive constant that depends on the shape of the vehicle and the density of the air.

The vehicle must move according to the following requirements. Tire traction limits acceleration so that $\dot{v}(t) \leq a^{\max}$. Note that there is no lower bound on the acceleration. The vehicle cannot move backwards and must stay within the speed limit, and so $0 \leq v(t) \leq v^{\max}$. The final velocity of the vehicle must satisfy $v(T) \leq v^{\text{final}}$.

We will use period $h > 0$ and sample position according to $x_i = x(ih)$, and similarly for velocity, acceleration and kinetic energy. The vehicle dynamics $\dot{x}(t) = v(t)$ and $\dot{v}(t) = a(t)$ are then discretized according to

$$x_{i+1} = x_i + \frac{h}{2}(v_i + v_{i+1}), \quad v_{i+1} = v_i + ha_i$$

and the rate of change of kinetic energy is discretized according to

$$\frac{1}{h}(k_{i+1} - k_i) = p_i - p_i^{\text{brake}} - p_i^{\text{loss}}$$

We would like to minimize the total energy used, which is discretized as

$$E = h \sum_{i=0}^{n} p_i$$

where $T = nh$. The parameters are

$$m = 10 \quad x^{\text{final}} = 10 \quad v^{\text{final}} = 1 \quad v^{\max} = 10 \quad a^{\max} = 2 \quad c^{\text{loss}} = 2 \quad h = 0.1 \quad T = 5$$

(a) Formulate this problem as an optimization problem with variables p_i, x_i, v_i, k_i (and others if necessary) for $i = 0, \ldots, n$. If this problem is not convex, explain briefly why.

(b) By relaxing the energy constraint $k(t) = \frac{1}{2}mv(t)^2$ to

$$k(t) \geq \frac{1}{2}mv(t)^2$$

state a convex optimization problem whose solution provides an optimal trajectory x, v, p, and k for part (a). Explain why the relaxation is tight. By tight, we mean that the solution to your problem has the same optimal value as that of part (a).

(c) Carry out your method from part (*b*). Report the optimal value of the total energy E. Plot the position x, velocity v and power used p of the vehicle as functions of time.

16.8 *Well that was a bit roundabout.* You're late for the last lecture of Convex Optimization and you need to get the lecture hall. You get on your bike, and proceed directly to class.

At time t the bike has position $x(t) \in \mathbf{R}^2$, velocity $v(t) \in \mathbf{R}^2$, and acceleration $a(t) \in \mathbf{R}^2$. You start at position $x(0) = x^{\text{initial}}$ and finish at time T with $x(T) = x^{\text{final}}$. Your initial velocity is $v(0) = v^{\text{initial}}$.

We will use period $h > 0$ and sample position according to $x_i = x(ih)$, and similarly for velocity and acceleration. Fortunately your bicycle is a point mass, and so the vehicle dynamics are $\dot{x}(t) = v(t)$ and $\dot{v}(t) = a(t)$. These are then discretized according to

$$\begin{aligned} x_{i+1} &= x_i + \tfrac{h}{2}(v_i + v_{i+1}) \\ v_{i+1} &= v_i + h a_i \end{aligned}$$

Despite your desire to arrive at class on time, you cycle somewhat leisurely, avoiding unnecessary exertion. So you choose to minimize

$$J = h \sum_{i=0}^{n} \|a_i\|_2^2$$

where $T = nh$. We have

$$x^{\text{initial}} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}, \quad x^{\text{final}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \quad v^{\text{initial}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad T = 12, \quad h = 0.1.$$

(a) Find and plot the optimal trajectory of the bicycle. Report the optimal value J.

(b) Unfortunately, somebody has built a roundabout in the way. The roundabout is a disk of radius 1 centered at the origin

$$R = \{x \in \mathbf{R}^2 \mid \|x\|_2 \leq 1\}$$

The constable observing your path advises you that he fears that your trajectory has the unfortunate property of failing to correctly circumnavigate the roundabout.

Unfortunately, the constraint that you should avoid the roundabout is not convex. After considering this, you arrive at a new strategy. Let the previous solution be x^{prev}. You construct a new optimization problem, where at each time step i you add the constraint that $c_i^T x_i \geq 1$, where

$$c_i = x_i^{\text{prev}} / \|x_i^{\text{prev}}\|_2.$$

Give a brief interpretation of these constraints. Solve the optimization problem again, with these new constraints. Plot the optimal trajectory and report the optimal cost.

(c) Repeat part (b) until the trajectory converges. Plot the final trajectory along with the the trajectories from part (a),(b) and the roundabout R. Note that each optimization only uses constraints generated by the previous solution. What is the final cost J achieved?

16.9 *Path planning with contingencies.* A vehicle path down a (straight, for simplicity) road is specified by a vector $p \in \mathbf{R}^N$, where p_i gives the position perpendicular to the centerline at the point ih meters down the road, where $h > 0$ is a given discretization size. (Throughout this problem, indexes on N-vectors will correspond to positions on the road.) We normalize p so $-1 \leq p_i \leq 1$ gives the road boundaries. (We are modeling the vehicle as a point, by adjusting for its width.) You are