Module 01
Course Syllabus, Scope, and Intro to Optimization

Dr. Ahmad F. Taha

CE 5999-02 Special Topics — Intro to Optimization

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VANDERBILT UNIVERSITY

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Course Instructor: Background & Interests

Background

- Born and raised in Beirut, Lebanon
- Finished my Ph.D. in ECE from Purdue University in August 2015
- Undergraduate education: American University of Beirut — Class of 2011, B.E., ECE
- Assistant Professor, ECE Department @ UTSA, August, 2015—August 2021
- Associate Professor, CEE Department @ Vandy, August 2021—currently

My Ultimate Objective

Understand how complex systems operate and utilize this knowledge to create tools & optimization and control algorithms to solve system-level challenges

Module 01 Outline

1. You will tell me about yourselves: careers, objectives, education
2. Course syllabus and expectations
3. Course outline
4. The fun stuff starts — we will introduce optimization and define the class scope

Part I — Your Turn to Introduce Yourselves! ☺

Part II — Course Syllabus and Outline

Course webpage & Communication

Course Page:

- Vanderbilt Brightspace: https://brightspace.vanderbilt.edu
- I might add the course content to my webpage—we'll see
- *Email is the best form of communication!*

Office Hours:

- Tuesdays and Thursdays, 11:00 – 12:30
- Or by appointment

Course Description

Introduction to optimization theory and methods, with applications in machine learning, systems control, and urban infrastructure design. In particular, this course offers a first look at optimization problem formulations, mathematical methods to solve optimization problems, and specifically convex optimization. Majority of machine learning, optimal control, and infrastructure (energy, water, transportation systems) design problems are posed as optimization problems, and this course presents the basics needed to understand these multidisciplinary engineering areas. Topics covered include: nonlinear unconstrained optimization, linear programming, nonlinear constrained optimization, various algorithms and search methods for optimization, and their analysis. Examples from various engineering applications and machine learning are given.

Main References

- No textbook is required for the class [PLEASE DO NOT BUY ANYTHING]
- But we will be referring to a book
- Lecture notes will be provided as handouts or presentation slides
- You may need to consult the following, mostly, free texts:
  - S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. YouTube videos for the class: https://www.youtube.com/watch?v=McLq1hEq3UY and book webpage: http://web.stanford.edu/~boyd/cvxbook/.
- Research papers

Course Objectives & Expected Outcomes

- This course is designed for graduate/undergraduate students who are interested in learning about optimization
- Course includes a wide range of topics related to convex optimization
- At the end of the semester, students are expected to have a good understanding of the basic principles governing engineering optimization

Prerequisites

An undergraduate-level understanding of:

- Multi-variable calculus
- Linear algebra
- Basics related to the aforementioned topics will be covered in the first two weeks of classes
- Most importantly: the will to learn [cheesy but true]

Grading Policy

- Homework assignments and quizzes (25%)
- Midterm exam (30%)
- Final exam (40%)
- Attendance and instructor evaluation (5%)

Course Grade Cutoffs

- A$-$, A, A$+$: 85–100
- B$-$, B, B$+$: 70–84
- C$-$, C, C$+$: 55–69
- D$-$, D, D$+$: 40–54
- F: $\leq$ 39

Programming Tools

- MATLAB will be required for homework assignments
- Students can obtain the discounted student version of MATLAB
- CVX, a toolbox in Matlab, will also be required and the first homework will walk you through installation and testing
- It's encouraged to use LaTeX for homework assignments (honestly, there's no good reason not to!)

Class Policies

- Regular attendance
- Emailing me
- Showing up early
- Smartphone breaks
- Late submission policy
- Changes to the syllabus

Tentative Class Schedule

Part I — Optimization Class Review & Background .... ≈ 1 class
- Course introduction & syllabus, prerequisites, major applications, course overview

Part II — Mathematical Background and Intro to Optimization .... ≈ 2–3 class
- Review of needed mathematical background

Part III — Convex Sets and Convex Functions .... ≈ 3–4 classes
- Description of what makes a convex function/set a convex one

Part IV — Convex Optimization Classes .... ≈ 3–4 classes
- Classes of various optimization problems and their difficulty

Part V — Duality .... ≈ 2–3 classes
- Introduction to a core principle in optimization: duality

Part VI — Optimization Algorithms .... ≈ 3–4 classes
- Solving optimization problems using mainstream algorithms

Part VII — Relaxation of Nonconvex Problems .... 2–3 class
- Transforming difficult optimization problems into easier ones to solve

Part VIII — Optimization Under Uncertainty .... ≈ 2–3 classes
- Optimization algorithms to solve problems with uncertain variables/parameters

Part IX — Applications .... ≈ 6–7 classes
- Water systems, power networks, machine learning, transportation, etc...

Part III — Introduction to Optimization

In this module

- What is an optimization problem?
- What is a solution of an optimization problem?
- Local vs global minimum
- The notion of iterative algorithms
- Easy vs difficult problems, or why we care about convex optimization
- Examples
- Contents of this course

Optimization in words

Minimize a cost function or maximize a profit function

Subject to some constraints on the decision variables

History of Optimization

- 300 BC: Euclid considers the minimal distance between a point a line
- 1636: Fermat shows that at extreme point, function derivative vanishes
- 1657: Fermat shows that light travels between two points in minimal time
- 1600s: Newton and von Leibniz create mathematical analysis that forms the basis of calculus of variations[1]
- 1740: Euler publishes general theory of calculus of variations
- 1806: Legendre presents the least square method
- 1847: Cauchy presents the gradient method
- 1800s: Convexity concepts are created
- 1917: Hancock publishes the first text book on optimization, *Theory of Minima and Maxima*
- 1947: Dantzig presents the Simplex method for solving linear programs
- 1951: Kuhn/Tucker reinvent optimality conditions for nonlinear pbms
- 1950s: Optimal control theory is created
- 1984: Karmarkar's efficient interior point method to solve opt pbms

[1] The calculus of variations is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of functionals.

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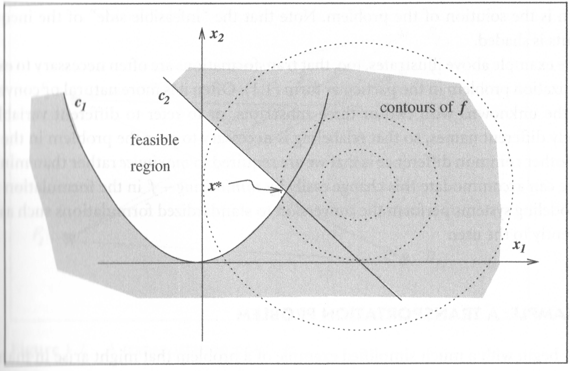
A simple example

$$\begin{aligned} \text{minimize} \quad & (x_1-2)^2+(x_2-1)^2 \\ \text{subject to} \quad & x_1^2-x_2 \le 0 \\ & x_1+x_2-2 \le 0 \end{aligned}$$



Picture from Nocedal-Wright, Numerical Optimization

Notations

- $\mathbb{R}$: The set of real numbers
- $\mathbb{R}_+$: The set of nonnegative numbers; $\mathbb{R}_{++}$: The set of positive numbers
- $\mathbb{R}^n$: The set of vectors with n entries (sometimes called the n-dimensional space)
- The elements of $\mathbb{R}^n$ will be called *vectors* or *points*
- Use column vector to write the entries: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- We will use figures in $\mathbb{R}^2$ and $\mathbb{R}^3$ as much as possible to illustrate concepts in $\mathbb{R}^n$
- x, a,... will be vectors or real numbers; it will always be clear, e.g., $x \in \mathbb{R}^n$, $a \in \mathbb{R}$
- $f(x) : \mathbb{R}^n \to \mathbb{R}$ means a function that takes an argument with n entries and returns a real number, e.g., $f(x) = e^{x_1 + \ldots + x_n}$

What is an optimization problem?

Minimization of a function subject to constraints on its variables.

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m & (\textbf{inequality constraints}) \\ & h_i(x) = 0, \quad i = 1, \ldots, p & (\textbf{equality constraints}) \end{array}$$

- $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a vector of **unknowns**, also called **variables**
- $f_0(x) : \mathbb{R}^n \to \mathbb{R}$ is the **objective function**, also called **cost function**
- $g_i(x) : \mathbb{R}^n \to \mathbb{R}$ and $h_i(x) : \mathbb{R}^n \to \mathbb{R}$ are the **constraint functions**

Solution of the optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{aligned}$$

We want to find a point x^* that minimizes $f_0(x)$ and *at the same time* satisfies the constraints $g_i(x) \leq 0$ for all $i = 1, \ldots, m$, and $h_i(x) = 0$, for all $i = 1, \ldots, p$.

The set of points that satisfy the constraints is called the **feasible set**.

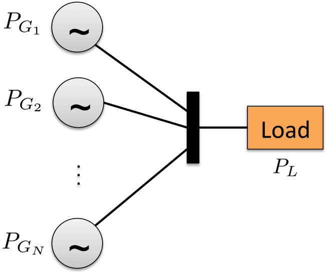
$$\mathcal{F} = \{x \in \mathbb{R}^n \text{ such that } g_i(x) \leq 0 \ (i = 1, \ldots, m) \text{ and } h_i(x) = 0 \ (i = 1, \ldots, p)\}$$

So the solution x^* of the problem must satisfy

$$f_0(x^*) \leq f_0(x) \text{ for all } x \text{ in the set } \mathcal{F}.$$

Motivating example from power engineering

- Suppose N power generation units serve a given load P_L (e.g., a city)
- The power output of unit i is P_{G_i} MW
- The cost of operating unit i is $C_i(P_{G_i})$ $/h

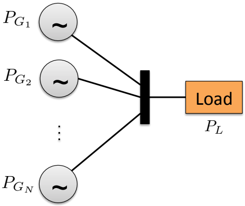


The economic dispatch problem

Given the cost functions $C_i(P_{G_i})$ and the load P_L, find the most economically generated power output.

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} C_i(P_{G_i}) \\ \text{subj. to} \quad & \sum_{i=1}^{N} P_{G_i} = P_L \\ & P_{G_i} \geq 0, \; i = 1, \ldots, N \end{aligned}$$

Economic dispatch is solved every 5-15 minutes in modern power grids.



Numerical example

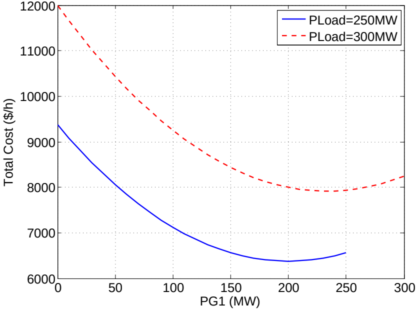
Two units: $C_1(P_{G_1}) = 0.025P_{G_1}^2 + 20P_{G_1}$ and $C_2(P_{G_2}) = 0.05P_{G_2}^2 + 25P_{G_2}$.

Comparison: Optimal solution vs. sharing the load equally between the two generators

P_L (MW)	$P_{G_1}^*$	$P_{G_2}^*$	Cost ($/h)	$P_{G_1} = \frac{P_L}{2}$	$P_{G_2} = \frac{P_L}{2}$	Cost ($/h)
250	200	50	6,375	125	125	6,797
300	233.3	66.7	7,917	150	150	8,438

The need for algorithms

In this example, we can read the solution by plotting $C_1(P_{G_1}) + C_2(P_L - P_{G_1})$ versus P_{G_1}.



What if we had hundreds of units?

We need **systematic methods** to solve optimization problems.

Modeling languages and solvers

Modeling language

- Accept the problem in a format similar to its mathematical form (minimize/subj. to)
- Then convert the problem to a format acceptable for a solver, and call the solver
- Easier and shorter code, but extra time is required to parse the problem

Solver

- Solves a specific type of problem (e.g., linear program) given the problem data
- Problem must be brought into a standard form (or in general, a form that the solver accepts)

We'll look at solving the economic dispatch using CVX (modeling language) and Matlab's quadprog (solver)

$$\begin{aligned} \text{minimize} \quad & 0.025P_{G_1}^2 + 20P_{G_1} + 0.05P_{G_2}^2 + 25P_{G_2} \\ \text{subject to} \quad & P_{G_1} + P_{G_2} = 250 \\ & P_{G_1} \geq 0,\ P_{G_2} \geq 0 \end{aligned}$$

Solvers

- Examples: MATLAB's linprog, quadprog
  https://www.mathworks.com/help/optim/ug/linprog.html
  https://www.mathworks.com/help/optim/ug/quadprog.html
- SeDuMi (http://sedumi.ie.lehigh.edu/)
- SDPT3 (http://www.math.nus.edu.sg/~mattohkc/sdpt3.html)
- MOSEK (https://www.mosek.com/)
- GUROBI (https://www.gurobi.com/)
- CPLEX (https://www.ibm.com/analytics/cplex-optimizer)

Modeling languages

- CVX (http://cvxr.com/cvx/)
- YALMIP (https://yalmip.github.io/)
  - See all solvers that YALMIP can call: https://yalmip.github.io/allsolvers/
- GAMS (https://www.gams.com/)
- AIMMS, AMPL, CPLEX Optimization Studio, Google OR-Tools

Economic dispatch as quadratic program

- Economic dispatch can be written as a quadratic program that has general form

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^T H x + f^T x \\ \text{subject to} \quad & Ax \preceq b \\ & A_{\text{eq}}x = b_{\text{eq}} \end{aligned}$$

- where H, A, A_{eq} are matrices and f, b, and b_{eq} are vectors
- The two sides of the constraint $Ax \preceq b$ are vectors
- The constraint means that an entry of the vector on the left is less than or equal to the corresponding entry of the vector on the right
- H, f, A,b, A_{eq}, b_{eq} are called *problem data* and are input to the solver

Writing the economic dispatch as a standard quadratic program

- Decision vector

$$x = \begin{bmatrix} P_{G_1} \\ P_{G_2} \end{bmatrix}$$

- Objective function

$$\frac{1}{2}x^T Hx + f^T x = \frac{1}{2}\begin{bmatrix} P_{G_1} \\ P_{G_2} \end{bmatrix}^T \underbrace{\begin{bmatrix} 2\cdot 0.025 & 0 \\ 0 & 2\cdot 0.005 \end{bmatrix}}_{H} \begin{bmatrix} P_{G_1} \\ P_{G_2} \end{bmatrix} + \underbrace{\begin{bmatrix} 20 \\ 25 \end{bmatrix}}_{f}^T \begin{bmatrix} P_{G_1} \\ P_{G_2} \end{bmatrix}$$

- Balance equation

$$A_{\text{eq}}x = b_{\text{eq}} \Leftrightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} P_{G_1} \\ P_{G_2} \end{bmatrix} = 250$$

- Nonnegativity constraints ($P_{G_1} \geq 0,\ P_{G_2} \geq 0$)

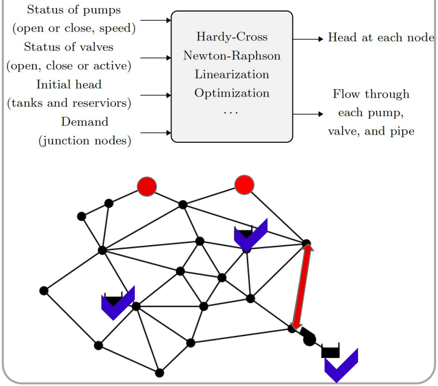
$$Ax \preceq b \Leftrightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} P_{G_1} \\ P_{G_2} \end{bmatrix} \preceq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- See Matlab codes using CVX and quadprog

Another example: Water flow problem in drinking networks

Water Flow Problem

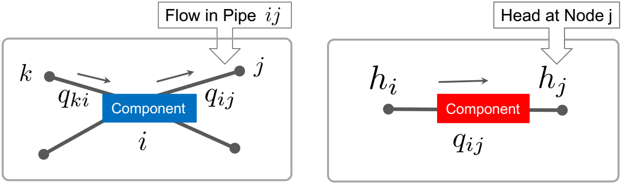
Solving for water flow and head (pressure) given water demand forecasts



WFP — 1

Components and principles in WDNs

Typical Topology



Components	Junctions, tanks, or reservoirs	Pipes, pumps, or valves
Principles	Conservation of mass	Conservation of energy
Effects	Linear equations	Nonlinear equations

All linear equations are in light blue, and all nonlinear equations are in red.

WFP — 2

Compo -nent	Hydraulic Modeling	Description
Junction	$\sum q^{\text{in}}(k) - \sum q^{\text{out}}(k) = d(k)$	Inflow – Outflow=Demand
Tank	$h^{\text{TK}}(k+1) = h^{\text{TK}}(k) + \alpha(\text{net flow})$	Tank Volume/Head Dynamics
Pipe	$\Delta h^{\text{P}}(k) = Rq(k)\,\lvert q(k)\rvert^{\mu-1}$	Head loss in pipes due to friction
Pump	$\Delta h^{\text{M}}(k) = -s^2(k)\left(h_0 - rq^{\nu}(k)s^{-\nu}(k)\right)$	Head gain through pump control
Valve	$\Delta h^{\text{W}}(k) = oRq(k)\lvert q(k)\rvert^{\mu-1}$	Head loss when water flows through a valve

WFP as a feasibility problem

Notation

Symbol	*Description*
$\boldsymbol{h}^{\mathrm{J}}$, $\boldsymbol{h}^{\mathrm{R}}$, $\boldsymbol{h}^{\mathrm{TK}}$	Heads at junctions, reservoirs, tanks
$\boldsymbol{q}^{\mathrm{P}}$,$\boldsymbol{q}^{\mathrm{M}}$,$\boldsymbol{q}^{\mathrm{W}}$	Flows in pipes, pumps, valves
$\boldsymbol{d}$	Demands at junctions

Differential Algebraic Equations (DAE)

Tank Dynamic: $\boldsymbol{h}^{\mathrm{TK}}(k+1) = \boldsymbol{h}^{\mathrm{TK}}(k) + \boldsymbol{E}^{\mathrm{TK}}\boldsymbol{q}^{\mathrm{P}}(k)$

Mass Balance: $\boldsymbol{d}(k) = \boldsymbol{E}_q \boldsymbol{q}(k)$

Energy Balance: $\boldsymbol{h}(k) = \boldsymbol{E}_h \boldsymbol{\Phi}(\boldsymbol{x}(k))$

Lumped variables

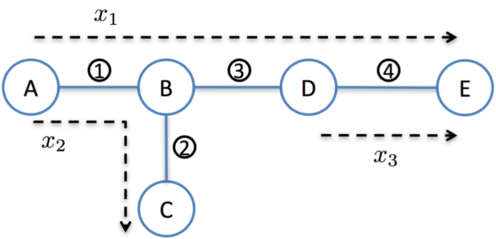
$$\boldsymbol{q}(k) \triangleq \left\{\boldsymbol{q}^{\mathrm{P}}(k), \boldsymbol{q}^{\mathrm{M}}(k), \boldsymbol{q}^{\mathrm{W}}(k)\right\}$$
$$\boldsymbol{h}(k) \triangleq \left\{\boldsymbol{h}^{\mathrm{J}}(k), \boldsymbol{h}^{\mathrm{R}}(k), \boldsymbol{h}^{\mathrm{TK}}(k)\right\}$$
$$\boldsymbol{x}(k) \triangleq \{\boldsymbol{h}(k), \boldsymbol{q}(k)\}$$

Box Constraints

$$\boldsymbol{x} \in \left[\boldsymbol{x}^{\min}, \boldsymbol{x}^{\max}\right]$$

Another example: Flow control in communication networks

- Communication network: Graph with L links
- S origin-destination pairs (sessions, flows); rate of session s is x_s (Mbps)
- Capacity of link ℓ is c_ℓ
- Set of sessions using link ℓ is $\mathcal{S}(\ell)$



Example: $S = 3$, $L = 4$, Links: AB, BC, BD, DE (indexed by circled numbers)
$\mathcal{S}(AB) = \{1, 2\}$, $\mathcal{S}(BC) = \{2\}$, $\mathcal{S}(BD) = \{1\}$, $\mathcal{S}(DE) = \{1, 3\}$

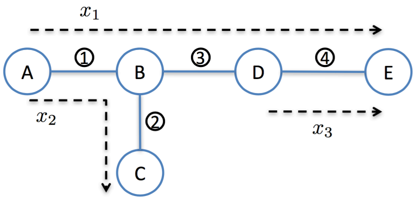
Flow control optimization problem

- Utility function $u_s(x_s)$ models happiness or income from supporting flow s at rate x_s

$$\begin{aligned} \text{maximize} \quad & \sum_{s=1}^{S} u_s(x_s) \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}(\ell)} x_s \leq c_\ell, \quad \ell = 1, \dots, L \\ & x_s \geq 0, \quad s = 1, \dots, S \end{aligned}$$

Algorithms for solving this problem have been the basis for improving the performance of congestion control in the Transmission Control Protocol (TCP) of the Internet.

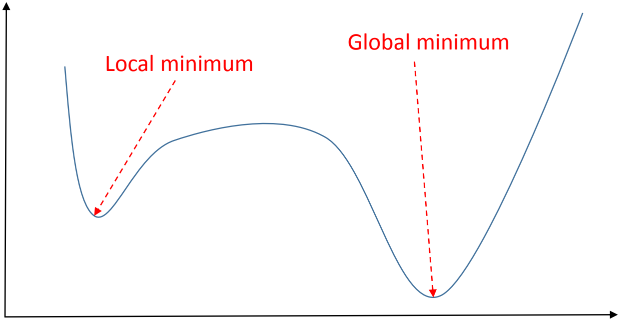
Flow control formulation example



$$
\begin{aligned}
\text{maximize} \quad & u_1(x_1) + u_2(x_2) + u_3(x_3) \\
\text{subject to} \quad & x_1 + x_2 \leq c_{AB} \\
& x_2 \leq c_{BC} \\
& x_1 \leq c_{BD} \\
& x_1 + x_3 \leq c_{DE} \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{aligned}
$$

Solving optimization problems

Iterative algorithm: Start with an initial guess $x^{(0)}$, then find the next iterate $x^{(1)}$, etc.
Is the sequence $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ going to converge to x^*?
Answer is provided by *convergence analysis*.



Difficult versus easy problems

General optimization problems are very difficult to solve

- May need very long computation time
- Find a global minimum or another point?

Convex optimization problems can be solved efficiently and reliably

- A local minimum is always a global minimum for a convex optimization problem (proof later in the class)
- The computational effort to solve a problem (e.g., number of iterations, or number of additions and multiplications required) versus the size of the problem scales better in the case of convex problems than in general (nonconvex) problems
- It is possible to have a *certificate of optimality* in convex optimization problems, that is, we can have a criterion to check whether a given point is indeed the optimal solution and get a yes or no answer

Convex versus nonconvex is the dividing line between easy and difficult

- Convex optimization problems may not be easy to recognize
- Some engineering problems are nonconvex; convex optimization methods can provide approximate solution

Convex optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{aligned}$$

- Functions $f_i(x)$, $i = 0, 1 \ldots, m$, are convex

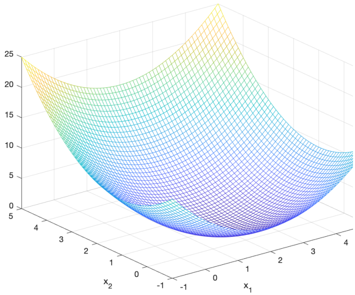
$$f_i(\alpha x + (1-\alpha)y) \le \alpha f_i(x) + (1-\alpha) f_i(y) \text{ for all } x, y, \text{and } \alpha \in (0,1)$$

- Functions $h_i(x)$, $i = 1 \ldots, p$, are linear

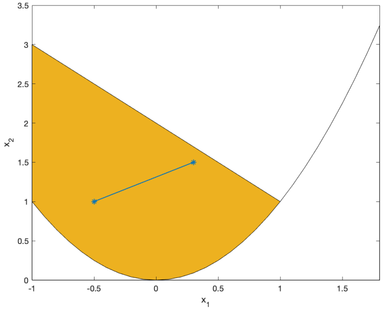
$$h_i(x) = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + b_i$$

Convex optimization problem: Geometric definition

- Objective function is convex: It has the shape of a cup ∪
- The feasible set is a convex set: For **any** two points inside the feasible set, the line segment connecting these points *remains inside the set*
- Consider the example in the beginning of the lecture



Objective function



Feasible set; and two points inside this set with the line segment connecting them

Applications

Communications and Networking

1. Power control in cellular networks
2. Capacity of parallel Gaussian channels and waterfilling
3. Optimal flow control and optimal routing

Energy, water, transportation

1. Economic dispatch,
2. Optimal power flow, optimal water flow, electricity markets, ramp metering
3. State estimation, water quality, traffic control

Signal Processing and Machine Learning

1. Least-squares, regression models, sparsity regularizations
2. Linear classification
3. Recommender systems: How does Netflix recommend movies?
4. Neural network training

Control

1. Optimal control of discrete-time linear systems
2. Numerous controllers are designed using Linear Matrix Inequalities

Aerospace and Navigation

1. Position, velocity, and time estimation by GPS

Contents (1)

Part 1: Learn to recognize convex optimization problems

1. Convex sets (Chapter 2 in textbook)
2. Convex functions (Chapter 3 in textbook)
3. Convex optimization problems (Chapter 4 in textbook)—includes linear programs, quadratic programs, ...

Part 2: Suppose we are given an optimization problem, and a specific point x with the claim that it is the solution. How can we check if x is indeed the solution?

1. Optimality conditions (Chapter 4)
2. Lagrangian duality (Chapter 5)

Contents (2)

Part 3: Iterative algorithms to find a solution

1. Gradient methods and Newton's method (Chapter 9)
2. Interior-point methods (Chapter 11)
3. Multiplier methods and the alternating-direction method of multipliers (ADMM) (slides)
4. Subgradient methods (slides)

Part 4: Extensions and more advanced topics

1. Decompositions and distributed optimization (slides)
2. Stochastic programming (slides)
3. Relaxations and approximations for nonconvex problems (slides)

Applications will be spread throughout the course.

Questions And Suggestions?

Thank You!

Please visit

https://lab.vanderbilt.edu/taha/

IFF you want to know more ☺