Lecture 11: SDP Relaxation and Randomized Methods for 0-1 Quadratic Program

(3 units)

Outline

- SDP problem
- Binary quadratic program and maximum cut problem
- SDP relaxation via Lagrangian dual
- SDP relaxation via lifting and rank relaxation
- Goemanns and Williamson's bound and randomized scheme
- Nesterov's SDP bound

SDP Problem

- Example of Linear program and semidefinite program (SDP):

$$
(LP) \qquad \begin{aligned} & \min 2x_1 + x_2 + x_3 \\ & \text{s.t. } x_1 + x_2 + x_3 = 1 \\ & \qquad (x_1, x_2, x_3) \geq 0. \end{aligned}
$$

$$
(SDP) \qquad \begin{aligned} & \min 2x_1 + x_2 + x_3 \\ & \text{s.t. } x_1 + x_2 + x_3 = 1 \\ & \qquad \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0. \end{aligned}
$$

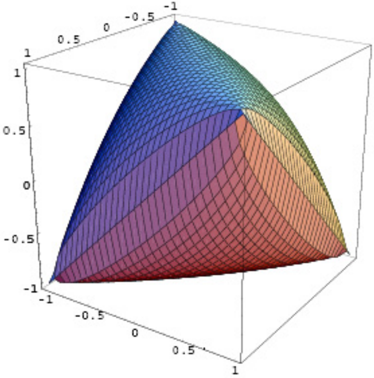


Figure: Set of 3×3 positive semidefinite matrices with unit diagonal

General form of SDP problem

- General form of SDP problem:

$$
\begin{aligned}
(SDP) \qquad & \min C \bullet X \\
& \text{s.t. } A_i \bullet X = b_i, \; i = 1, \ldots, m, \\
& \quad\; X \succeq 0,
\end{aligned}
$$

  where C, A_i are given $n \times n$ symmetric and b_is are given scalars, and

$$
A \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij} = Tr(A^T x).
$$

- The dual of (SDP) is

$$
\begin{aligned}
(SDD) \qquad & \max b^T y \\
& \text{s.t. } \sum_{i=1}^{m} y_i A_i \preceq C,
\end{aligned}
$$

  where $y \in \mathbb{R}^m$.

- Or equivalently

$$(SDD) \qquad \min b^T y$$
$$\text{s.t. } \sum_{i=1}^{m} y_i A_i + S = C,$$
$$S \succeq 0.$$

- Strong duality: $v(SDP) = v(SDD)$ if (SDP) or (SDD) is strictly feasible.
- The SDP interior-point algorithm finds an ϵ-approximate solution where solution time is linear in $\log(1/\epsilon)$ and polynomial in m and n.
- SDP solvers and software based on MatLab: SeDuMi, CVX ($n \leq 1000$).

Binary quadratic optimization and max-cut problem

- Binary quadratic optimization:

$$\begin{aligned}&\min x^T Qx\\&\text{s.t. } x \in \{-1, 1\}^n.\end{aligned}$$

  This is a well known NP-hard problem. It is NP-hard even if the matrix Q is positive definite, since $x^T Qx = x^T (Q + diag(\lambda))x - e^T \lambda$.
- The Boolean constraints can be expressed using quadratic equations:

$$x_i^2 - 1 = 0 \Leftrightarrow x_i \in \{-1, 1\}.$$

- An equivalent problem:

$$\begin{aligned}&\min x^T Qx\\&\text{s.t. } x_i^2 = 1,\ i = 1, \ldots, n.\end{aligned}$$

- Example. The maximum cut (MAXCUT) problem is to find a partition of the nodes of a graph $G = (V, E)$ into two disjoint sets V_1 and V_2 ($V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$) in such a way to maximize the weights of edges that have one endpoint in V_1 and the other in V_2. Let w_{ij} be the weight corresponding to the (i, j) edge, and is zero if the nodes i and j are not connected.
- Define:

$$y_i = -1 \Leftrightarrow i \in V_1, \quad y_i = 1 \Leftrightarrow i \in V_2.$$

The weight of the cut defined by $y \in \{-1, 1\}^n$ is:

$$\frac{1}{4} \sum_{i,j=1}^{n} w_{ij}(1 - y_i y_j)$$

$(i \in V_1, j \in V_2 \Rightarrow w_{ij}(1 - y_i y_j) = 2w_{ij},$
$i, j \in V_1 \text{ or } V_2 \Rightarrow w_{ij}(1 - y_i y_j) = 0).$

- MAXCUT problem can be written as:

$$\max \frac{1}{4} \sum_{i,j} w_{ij}(1 - y_i y_j)$$
$$\text{s.t. } y_i \in \{-1, 1\},\ i = 1, \ldots, n,$$

- The maximum cut problem is clearly equivalent to:

$$\min \sum_{i,j} w_{ij} y_i y_j, \quad \text{s.t. } y_i \in \{-1, 1\},\ i = 1, \ldots, n.$$

SDP relaxation via Lagrangian dual

- Lagrangian duality A general approach to obtain lower bounds on the value of general (non)convex minimization problems is to use Lagrangian duality. The original Boolean minimization problem can be written as:

$$(P) \qquad \begin{aligned} &\min x^T Q x \\ &\text{s.t. } x_i^2 = 1, \; i = 1, \ldots, n. \end{aligned}$$

- The Lagrangian function can be written as:

$$L(x, \lambda) = x^T Q x - \sum_{i=1} \lambda_i (x_i^2 - 1) = x^T (Q - \Lambda) x + Tr(\Lambda).$$

- The Lagrangian relaxation of (P) is

$$d(\lambda) = \min_{x \in \mathbb{R}^n} L(x, \lambda) = x^T (Q - \Lambda) x + Tr(\Lambda).$$

The dual of (P) is

$$\begin{aligned} &\max \; d(\lambda) \\ &\text{s.t. } \lambda \in \mathbb{R}^n. \end{aligned}$$

- $d(\lambda) > -\infty \Leftrightarrow$
  (i) $Q - \lambda \succeq 0$;
  (ii) $\exists \bar{x} \in \mathbb{R}^n$ such that $(Q - \Lambda)\bar{x} = 0$.
- Proof. $d(\lambda) > -\infty \Rightarrow Q - \lambda \succeq 0$. If, otherwise, there exists $\tilde{x} \neq 0$ such that $\tilde{x}^T(Q - \Lambda)\tilde{x} < 0$, then

  $$L(t\tilde{x}, \mu) = \frac{1}{2}t^2\tilde{x}^T(Q - \Lambda)\tilde{x} + Tr(\Lambda) \to -\infty, \qquad t \to +\infty.$$

  $Q - \Lambda \succeq 0$ and the KKT necessary condition $\Rightarrow (Q - \Lambda)\bar{x} = 0$.

- Conversely, if conditions (i)-(ii) hold, then $L(x, \lambda)$ is a convex function and $\bar{x}$ satisfies the KKT sufficient condition. Thus, $d(\lambda) = L(\bar{x}, \lambda) = Tr(\Lambda)$
- $d(\lambda) > -\infty \Rightarrow d(\lambda) = Tr(\Lambda)$.
- The dual problem can be written as an SDP problem:

$$\begin{aligned} &\max \ Tr(\Lambda) \\ &\text{s.t. } Q \succeq \Lambda, \\ &\qquad \Lambda \text{ diagonal}, \end{aligned}$$

or

$$\begin{aligned} &\max \ e^T \lambda \\ &\text{s.t. } Q - \mathrm{diag}(\lambda) \succeq 0. \end{aligned}$$

Lifting and rank relaxation

- Let $X = xx^T$ for $x \in \{-1, 1\}^n$. Then $X \succeq 0$, $X_{ii} = 1$, and X has rank one. Conversely, any matrix X with $X \succeq 0$, $X_{ii} = 1$, $rank X = 1$, then $X = xx^T$ for some $x \in \{-1, 1\}^n$.
- $x^T Qx = Tr(x^T Qx) = Tr(Qxx^T) = Tr(QX)$.
- As a consequence, the original problem can be exactly rewritten as:

$$\begin{aligned} \min \;& Tr(QX) \\ \text{s.t. } & X_{ii} = 1, \; i = 1, \dots, n \\ & X \succeq 0, \; rank(X) = 1. \end{aligned}$$

- For any $X \succeq 0$ and $diag(X) = e$,

$$rank(X) = 1 \Leftrightarrow \|X\| = n.$$

- Proof. By definition

$$\|X\|^2 = Tr(X^2) = \sum_{i=1}^{n} \lambda_i(X)^2 = \|\lambda(X)\|^2.$$

  For any $x \in \mathbb{R}^n$,

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} |x_i| = \|x\|_1$$

  with equality if and only if there is at most one nonzero x_i. So

$$\|X\| \leq \sum_{i=1}^{n} |\lambda_i(X)| = n$$

  with equality if and only $rank(X) = 1$.
- Dropping the rank one constraint results in a SDP relaxation

$$\begin{aligned} \min \ & Tr(QX) \\ \text{s.t. } & X_{ii} = 1, \ i = 1, \dots, n \\ & X \succeq 0. \end{aligned}$$

- It SDP dual is:

$$\begin{aligned} &\max \ e^T\lambda \\ &\text{s.t. } Q - \text{diag}(\lambda) \succeq 0. \end{aligned}$$

- A useful interpretation is in terms of a nonlinear lifting to a higher dimensional space. Indeed, rather than solving the original problem in terms of the n-dimensional vector x, we are instead solving for the $n \times n$ matrix X, effectively converting the problem from R^n to S^n, which has dimension $\left(\begin{array}{c} 2 \\ n+1 \end{array} \right)$.
- If we find an optimal solution X of the SDP that has rank one $\Rightarrow$ the original problem is solved.
- In general, it is not the case that the optimal solution of the SDP relaxation will be rank one. However, it is possible to use rounding schemes to obtain nearby rank one solutions. Furthermore, in some cases, it is possible to do so while obtaining some approximation guarantees on the quality of the rounded solutions.

Goemanns and Williamson's bound

- Basic questions:
  - Approximation guarantees: is it possible to prove general properties on the quality of the bounds obtained by SDP?
  - Feasible solutions: can we (somehow) use the SDP relaxations to provide not just bounds, but actual feasible points with good (or optimal) values of the objective?
- In their celebrated MAXCUT paper (JACM, 1995), Goemans and Williamson developed the following randomized method for finding a good feasible cut from the solution of the SDP:
  - Factorize X as $X = V^T V$, where $V = [v_1, \ldots, v_n] \in \mathbb{R}^{r \times n}$, where r is the rank of X.
  - Then $X_{ij} = v_i^T v_j$, and since $X_{ii} = 1$ this factorization gives n vectors v_i on the unit sphere in $\mathbb{R}^r$.
  - Now, choose a random hyperplane (passing through the origin) in $\mathbb{R}^r$, and assign to each variable x_i either $+1$ or -1, depending on which side of the hyperplane the point v_i lies.

- It turns out that this procedure gives a solution that, on average, is quite close to the value of the SDP bound. The random hyperplane can be characterized by its normal vector p, which is chosen to be uniformly distributed on the unit sphere.
- The rounded solution is given by $x_i = sign(p^T v_i)$. The expected value of this solution in $x^T W x$ can then be written as:

$$\begin{aligned} E_p[x^T W x] &= \sum_{i,j} w_{ij} E_p[x_i x_j] \\ &= \sum_{i,j} w_{ij} E_p[sign(p^T v_i) \cdot sign(p^T v_j)]. \end{aligned}$$

- Consider the plane spanned by v_i and v_j , and let θ_{ij} be the angle between these two vectors. Then,

  $$E_p[sign(p^T v_i) \cdot sign(p^T v_j)] = P_1 \times 1 + P_2 \times (-1) = 1 - \frac{2\theta_{ij}}{\pi}$$

  P_1=probability that both points are on the same side of the hyperplane=$1 - \frac{\theta_{ij}}{\pi}$.
  P_2=probability that they are on different sides=$\frac{\theta_{ij}}{\pi}$.
  (if $\theta_{ij} = \pi \Rightarrow v_i$ and v_j must be in different sides of any plane.)
- Thus

$$\begin{aligned} E_p[x^T W x] &= \sum_{i,j} w_{ij} \left(1 - \frac{2\theta_{ij}}{\pi}\right) \\ &= \sum_{i,j} w_{ij} \left(1 - \frac{2}{\pi} \arccos(v_i^T v_j)\right) \\ &= \frac{2}{\pi} \sum_{i,j} w_{ij} \arcsin X_{ij} \end{aligned}$$

- Notice that the expression is of course well defined, since if $X \succeq 0$ and has unit diagonal $\Rightarrow$ $|X_{ij}| \leq 1$.
- The objective of maximum cut is $\frac{1}{4}\sum_{i,j} w_{ij}(1 - y_i y_j)$. So the expected value of the cut is then:

$$
\begin{aligned}
c_{\mathrm{sdp-expected}} &= \frac{1}{4}\sum_{i,j}\left(1 - \frac{2}{\pi} w_{ij} \arcsin X_{ij}\right) \\
&= \frac{1}{4} \cdot \frac{2}{\pi}\sum_{i,j} w_{ij} \arccos X_{ij}.
\end{aligned}
$$

- On the other hand, the solution of the SDP gives an upper bound on the cut capacity equal to:

$$
c_{\mathrm{sdp-upperbound}} = \frac{1}{4}\sum_{i,j} w_{ij}(1 - X_{ij}).
$$

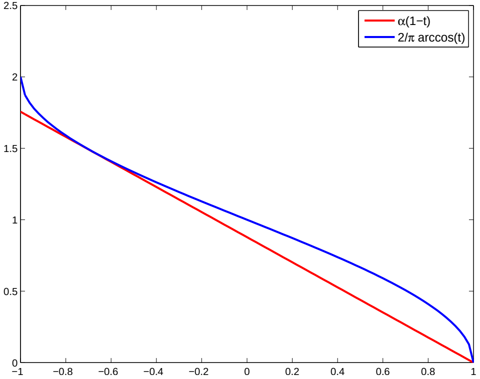
- Consider the problem of finding a constant α such that

$$\alpha(1-t) \leq \frac{2}{\pi}\,\mathrm{arccos}(t), \qquad \forall t \in [-1,1].$$

  This is

$$\alpha = \min_{t\in[-1,1]} \frac{2}{\pi}\frac{\mathrm{arccos}(t)}{1-t} = \min_{\theta\in[0,\pi]} \frac{2}{\pi}\frac{\theta}{1-\cos\theta}.$$

- It can be shown that $0.87856 < \alpha < 0.87857$.


α(1−t)
2/π arccos(t)
2.5
2
1.5
1
0.5
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

- Thus

$$c_{\text{sdp-upperbound}} \leq \frac{1}{4} \cdot \frac{1}{\alpha} \sum_{i,j} w_{ij} \arccos(X_{ij}) = \frac{1}{\alpha} c_{\text{sdp-expected}}.$$

- So far we have the following inequalities:

$$c_{\text{sdp-upperbound}} \leq \frac{1}{\alpha} c_{\text{sdp-expected}}$$
$$c_{\text{sdp-expected}} \leq c_{\max}$$
$$c_{\max} \leq c_{\text{sdp-upperbound}}$$

- Therefore

$$\boxed{0.878 \cdot c_{\text{sdp-upperbound}} \leq c_{\max}}$$

(approximation ratio for SDP bound)

$$\boxed{0.878 \cdot c_{\max} \leq c_{\text{sdp-expected}}}$$

(approximation ratio for feasible solution)

Nesterov's SDP bound

- In the MAXCUT problem, we are in fact maximizing the homogeneous quadratic form (omitting $\frac{1}{4}$):

$$x^T Ax = \sum_{i,j=1}^{n} w_{ij}(1 - x_i x_j) = \sum_{i=1}^{n}(\sum_{j=1}^{n} w_{ij})x_i^2 - \sum_{i,j=1}^{n} w_{ij} x_i x_j$$

  over $\{-1, 1\}^n$.
- Special properties of A:
  - $A \succeq 0$ (why?);
  - $A_{ij} \leq 0$ for all $i \neq j$;
  - $\sum_{j=1}^{n} A_{ij} = 0$.
- What happens if A is a general positive semidefinite matrix?
- Let $A \succeq 0$, consider the problem:

$$(P) \qquad \max_{x \in \{-1,1\}^n} x^T Ax.$$

- The SDP relaxation of (P) is

$$
(SDP) \qquad \begin{aligned} \max\ & A \bullet X \\ \text{s.t. } & diag(X) = e, \\ & X \succeq 0. \end{aligned}
$$

$\frac{v(SDP)}{v(P)} \geq 1.$

- Theorem (Nesterov (1998). Let $A \succeq 0$. Then

$$
\boxed{v(P) \leq v(SDP) \leq \frac{\pi}{2} v(P)}
$$

- Let X be the optimal solution to SDP. Let $X = V^T V$, where $V = (v_1, \ldots, v_n)$ with $v_i \in \mathbb{R}^r$. Let ξ be a Gaussian random vector with zero mean and covariance matrix X. Let $\zeta = sign(\xi)$. Then

$$\begin{aligned} v_{SDP-e} = E(\zeta^T A \zeta) &= \frac{2}{\pi} \sum_{i,j=1}^{n} A_{ij} \arcsin(X_{ij}) \\ &= \frac{2}{\pi} Tr(A \arcsin[X]) \\ &= \frac{2}{\pi} \langle A, \arcsin[X] \rangle, \end{aligned}$$

  where $|X_{ij}| \leq 1$ since $X \succeq 0$ and $X_{ii} = 1$.
- Schur product of matrices: Let $A = (a_{ij})$, $B = (b_{ij})$. Then $A \circ B = [a_{ij} b_{ij}]$ is called Schur product.

- Schur Product Theorem: Let $A, B \in \mathcal{S}_n^+$. Then $A \circ B \in \mathcal{S}_n^+$. In particular $A \succeq 0 \Rightarrow A^{\circ k} \succeq 0$.
- Proof. For $v \in \mathbb{R}^n$, we have $(A \circ B)v = Diag(A \cdot diag(v)B)$. So

$$\begin{aligned} v^T(A \circ B)v &= v^T Diag(A \cdot diag(v)B) \\ &= Tr(diag(v)A \cdot diag(v)B) \\ &= \langle diag(v)A \cdot diag(v), B\rangle \geq 0. \end{aligned}$$

$(A, B \in \mathcal{S}_n^+ \Rightarrow \langle A, B\rangle \geq 0$, why? Prove it.)

- Taylor expansion. For $t \in [-1, 1]$, we have

$$\arcsin(t) = t + \frac{1}{2}\frac{t^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{t^5}{5} + \cdots$$

- Since $|X_{ij}| \leq 1$, we have

$$\arcsin[X] = X + \frac{1}{2}\frac{X^{\circ 3}}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{X^{\circ 5}}{5} + \cdots$$

Since $X \succeq 0 \Rightarrow X^{\circ k} \succeq 0$, we get

$$\arcsin[X] \succeq X.$$

Again, since $X \succeq 0$, we obtain

$$\langle A, \arcsin[X] \rangle \geq \langle A, X \rangle.$$

- Thus,

$$v_{SDP-e} = \frac{2}{\pi}\langle A, \arcsin[X] \rangle \geq \langle A, X \rangle = \frac{2}{\pi} v(SDP).$$

Since $v_{SDP-e} \leq v(P) \leq v(SDP)$, we finally get

$$\frac{2}{\pi} v(P) \leq \frac{2}{\pi} v(SDP) \leq v_{SDP-e} \leq v(P) \leq v(SDP).$$

- Thus

$$v(P) \geq \frac{2}{\pi} v(SDP).$$

$$v(SDP - e) \geq \frac{2}{\pi} v(P).$$

$\frac{2}{\pi} \approx 0.6366.$