



*Electrical &
Computer
Engineering*

EE 1322: Intro to Electrical & Computer Engineering

Module 3: Mathematical Basics

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Why do we need math?

- Every major societal and engineering achievement in the past two centuries relied on mathematics
- Some of this math is very simply (multiplication and division, basic calc)
- But some relied on advanced math; examples:
 - ✓ - Google's PageRank algorithm => best search engine for 3+ decades
 - ✓ - SpaceX's technology
 - - Spam emails detection
 - ✓ - Cancer therapy
 - ✓ - Sequencing of genes
 - ✓ - Driverless cars
 - ...
- So, math is truly the cornerstone of good, impactful engineering
- Fighting the need to study math (and complaining about it) ain't making you happy
- Math is applied, useful, needed—not a useless thing you're forced to learn
 - Whoever tells you so, take their phone away and lock them out of their social media accounts

Math is very broad

- Roughly two areas of math (actually many more)
 - *Applied math*: study of math which the purpose of applying it to a problem
 - *Real math*: study of abstract theory that might have no application
- In ECE, we need applied math although studying real math helps you grow
- Applied math in engineering and ECE:
 - Linear algebra (vectors and spaces and what not)
 - Calculus (functions and how they evolve)
 - Statistics and probability
 - Numerical computing
- All of these topics are related to each other
 - Cannot study statistics without linear algebra and calculus, for example
- Good news: with practice you can become so good at it
 - All of these topics give you recipes; the excuse “I’m not good at it” is kinda lame :P
- Bad news: no bad news
- This module: lightweight intro to linear algebra and complex numbers

1. Vectors and linear algebra
2. Complex numbers
3. Basic calculus

Part 1:

Vectors and Linear Algebra

Vectors

- A vector is an ordered list of numbers
- written as

can be

$$\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{pmatrix}$$

$$x = \begin{bmatrix} 3 \\ \pi \\ 1.5 \end{bmatrix}$$

$$y = \begin{bmatrix} -7/2 \\ -\pi^2 \end{bmatrix}$$

$$\text{or } (-1.1, 0, 3.6, -7.2)$$

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

- Numbers in the list are the elements (entries, coefficients, components)
- Number of elements is the size (dimension, length) of the vector
- Vector above has dimension 4; its third entry is 3.6
- Vector of size n is called an n-vector
- Numbers are called scalars

x

$$\boxed{x=2} \rightarrow \underline{\underline{\text{scalar}}}$$

More on vectors



- We'll use symbols to denote vectors: $\vec{a}$, $\mathbf{v}$, $\mathbf{x}$, ...
- Other conventions: $\mathbf{a}$, $\mathbf{v}$, $\mathbf{x}$; or $\vec{a}$
- The i th element of an n -vector $\mathbf{a}$ is denoted a_i
- If $\mathbf{a}$ is vector above, $a_3 = 3.6$
- For an n -vector, indices run from $i = 1$ to $i = n$
- Two vectors $\mathbf{a}$ and $\mathbf{b}$ of the same size are equal if $a_i = b_i$ for all i
- Examples for a vector $\mathbf{x}$:

cash flow: x_i is payment in period i in us

audio: x_i is the acoustic pressure at sample time i
(sample times are spaced 1/44100 seconds apart)

features: x_i is the value of i th *feature* or *attribute* of an entity

customer purchase: x_i is the total \$ purchase of product i by a customer over some period

word count: x_i is the number of times word i appears in a document

$$\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \quad \begin{array}{l} \text{4-dim.} \\ \text{-vector} \end{array}$$

$x =$ $\begin{bmatrix} \text{blue} \\ \text{green} \\ \text{black} \\ \vdots \end{bmatrix}$

$x =$ $\begin{bmatrix} 2.5 \\ \vdots \end{bmatrix}$

More examples on vectors

a short document:

Word count vectors are used **in** computer based **document** analysis. Each entry of the **word** count vector is the **number** of times the associated dictionary **word** appears **in** the **document**.

a small dictionary (left) and word count vector (right)

word		3
in		2
number		1
horse	$X =$	0
the		4
document		2

NLP

dictionaries used in practice are much larger

What is a vector ?

$$X = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 4 \end{bmatrix}, X^T = [-1 \ -2 \ 3 \ 4]$$

- Again, a vector as a directed line segment in n-dimensions! (has “length” and “direction”)

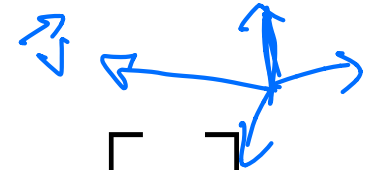
$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \leftarrow$$

- Basic idea: convert geometry in higher dimensions into algebra!

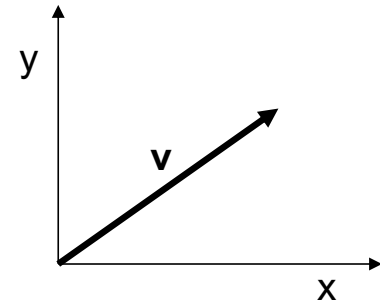
- Once you define a “nice” basis along each dimension: x-, y-, z-axis ...
- Vector becomes easy to define algebraically
- $\underline{\mathbf{v}} = [a \ b \ c]^T$

- The ‘T’ is called transpose
- Geometry starts to become linear algebra on vectors like $\mathbf{v}$!

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} : \text{column vector} \quad ; \quad X^T = [1 \ 2 \ 3] : \text{row vector}$$



$$\vec{\mathbf{v}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

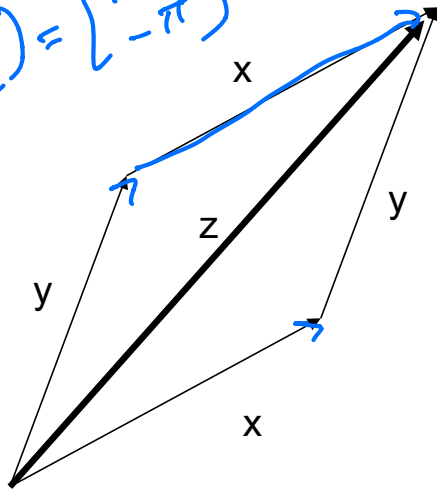


Vectors addition

$$\mathbf{x} + \mathbf{y} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \pi \end{pmatrix}$$
$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -\pi \end{pmatrix}$$

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 2 \\ \pi \end{pmatrix} + \begin{pmatrix} -1 \\ -\pi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$\mathbf{x} + \mathbf{y} = \mathbf{z}$
(use the head-to-tail method
to combine vectors)

Example:

$$\begin{pmatrix} \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix} \end{pmatrix}$$

Scaling vectors

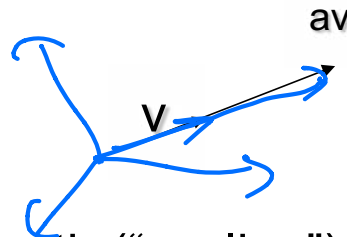
$$a = \pi \\ = 3.14 \dots$$

$$\underline{a\mathbf{v}} = a(\underline{x_1, x_2}) = (ax_1, ax_2)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$a\mathbf{v} = \begin{bmatrix} 2\pi \\ 7\pi \end{bmatrix}$$



Change only the length (“scaling”), but keep direction fixed.

Example:

$$\underline{a} \quad \mathbf{v} \\ (-2) \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Vectors: dot product : for vectors of same size (dimension)

$$x, y \quad x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = x \cdot y = \sum_{i=1}^n x_i y_i$$

$$\underline{A \cdot B = A^T B} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

$$\|A\|^2 = A^T A = aa + bb + cc$$

The magnitude is the dot product of a vector with itself

$$x \cdot x = x_1 x_1 + x_2 x_2 + \dots + x_n x_n = x_1^2 + x_2^2 + \dots + x_n^2 = \|x\|_2^2$$

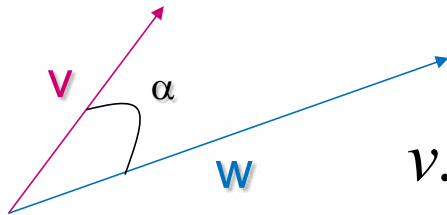
$$\underline{\underline{A \cdot B = \|A\| \|B\| \cos(\theta)}}$$

The dot product is also related to the angle between the two vectors



Inner products

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 5 = 2 + 8 + 15 = 25$$



$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 \cdot y_2$$

The inner product is a **SCALAR!**

2.5

$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = \|v\| \cdot \|w\| \cos \alpha$$

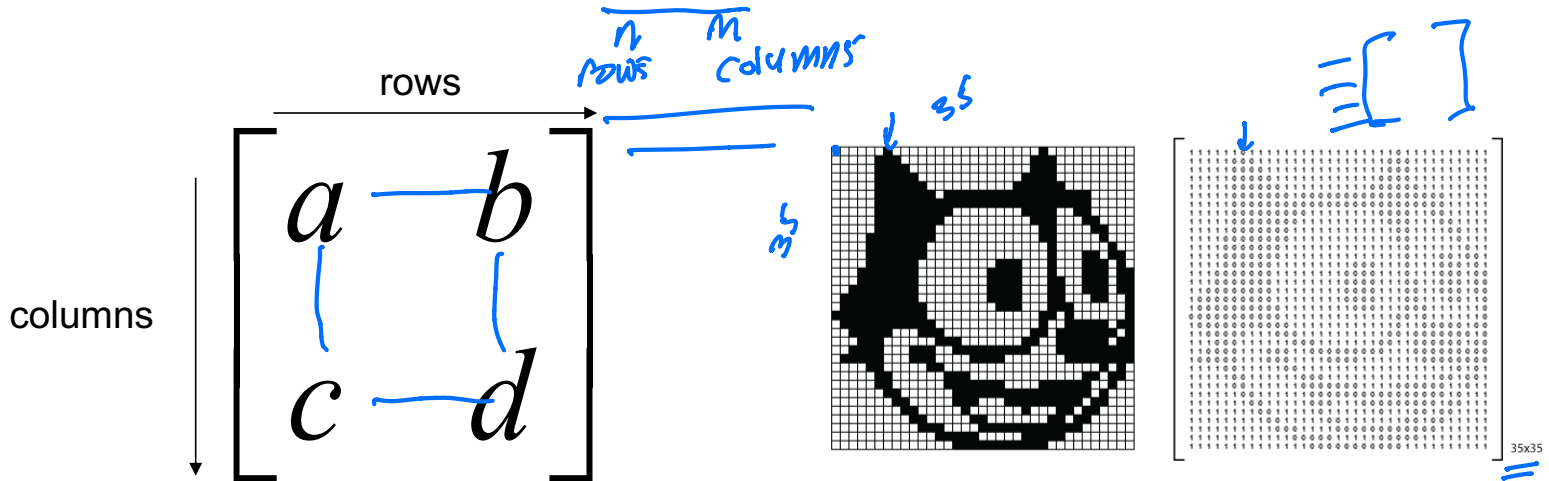
$$v \cdot w = 0 \Leftrightarrow v \perp w$$

If vectors **v**, **w** are “columns”, then dot product is **$w^T v$**

Example: $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = (-1)(1) + (2)(0) + (2)(-3) = -7$

What is a matrix?

- A matrix is a set of elements, organized into rows and columns
- For example, this matrix is a 2-by-2 matrix (two rows and two columns)



- Pictures are matrices and filters are operations on these matrices!

- Scalars, vectors, matrices, tensor

Handwritten examples of mathematical objects:

- 2 (Scalar)
- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (Vector)
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (Matrix)

matrices

- square ($n = m$)
- rectangular ($n < m$), ($n > m$)

Basic matrix operations

- Addition, Subtraction, Multiplication: creating new matrices (or functions)

$$\begin{bmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{bmatrix} + \begin{bmatrix} \underline{e} & \underline{f} \\ \underline{g} & \underline{h} \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \underline{ae+bg} & \underline{af+bh} \\ \underline{ce+dg} & \underline{cf+dh} \end{bmatrix}$$

Handwritten annotations: Blue arrows show row-by-column multiplication. Dimensions 2x2, 2x2, and 2x2 are written below the matrices. The result matrix is circled in blue.

Multiply each row
by each column

- If you want to give me a heart attack, add matrices of different dimensions. That will literally send me to the ER ☹️

Matrix times matrix

$$\mathbf{L} = \mathbf{M} \cdot \mathbf{N}$$

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$l_{12} = m_{11}n_{12} + m_{12}n_{22} + m_{13}n_{32}$$

- You cannot multiply matrices of arbitrary dimensions.
- The dimensions need to make sense.
- Examples: snapchat filters, tiktok videos (yes, videos are many images put together)

- Is $AB = BA$? Maybe, but maybe not!

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_A \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_B = \begin{bmatrix} \overbrace{ae+bg} & \dots \\ \dots & \dots \end{bmatrix} \quad \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_B \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_A = \begin{bmatrix} \overbrace{ea+fc} & \dots \\ \dots & \dots \end{bmatrix}$$

- Matrix multiplication AB : apply transformation B first, and then again transform using A !
- Heads up: multiplication is NOT commutative!
- We say matrices A and B commute if and only if $AB = BA$
 – Otherwise, they do not commute

Matrix operating on vectors

- Solving two equations with two unknowns example

unknowns: x, y

$$\begin{cases} (1) & 2x + y = 1 \\ (2) & -3x + 2y = 0 \end{cases}$$

unknowns
↓
 $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- This can be written as a matrix-vector equation!
- Matrix is like a function that transforms the vectors on a plane
- Matrix operating on a general point $\Rightarrow$ transforms x- and y-components
- System of linear equations*: matrix is just the bunch of coefficients!

- $x' = ax + by$
- $y' = cx + dy$



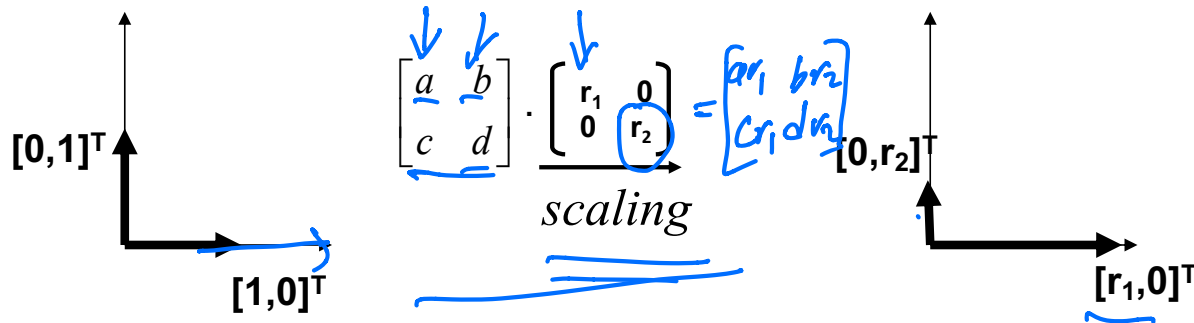
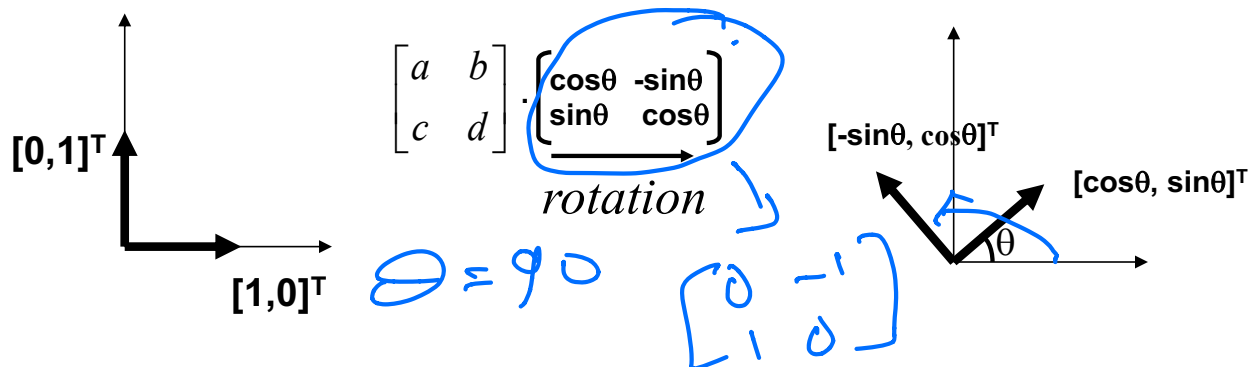
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$\downarrow$
 $\underline{Av = z}$
 $\nwarrow$

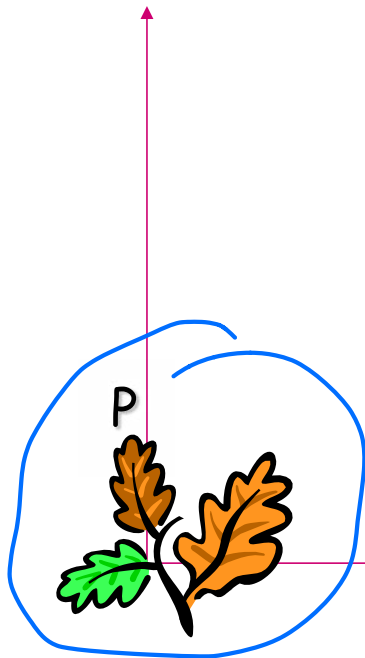
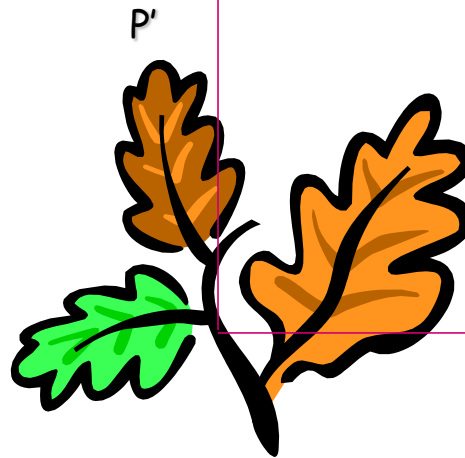
- In the above example, $a = 2$, $b = 1$, $x' = 1$, $c = -3$, $d = 2$, $y' = 0$

Matrices: scaling, rotation, identity

- Pure scaling, no rotation => “**diagonal** matrix” (note: x-, y-axes could be scaled differently!)
- Pure rotation, no stretching => “**orthogonal** matrix” **O**
- **Identity** (“do nothing”) matrix = unit scaling, no rotation!



Scaling



$$P = \begin{bmatrix} 100 \times 100 \end{bmatrix}$$

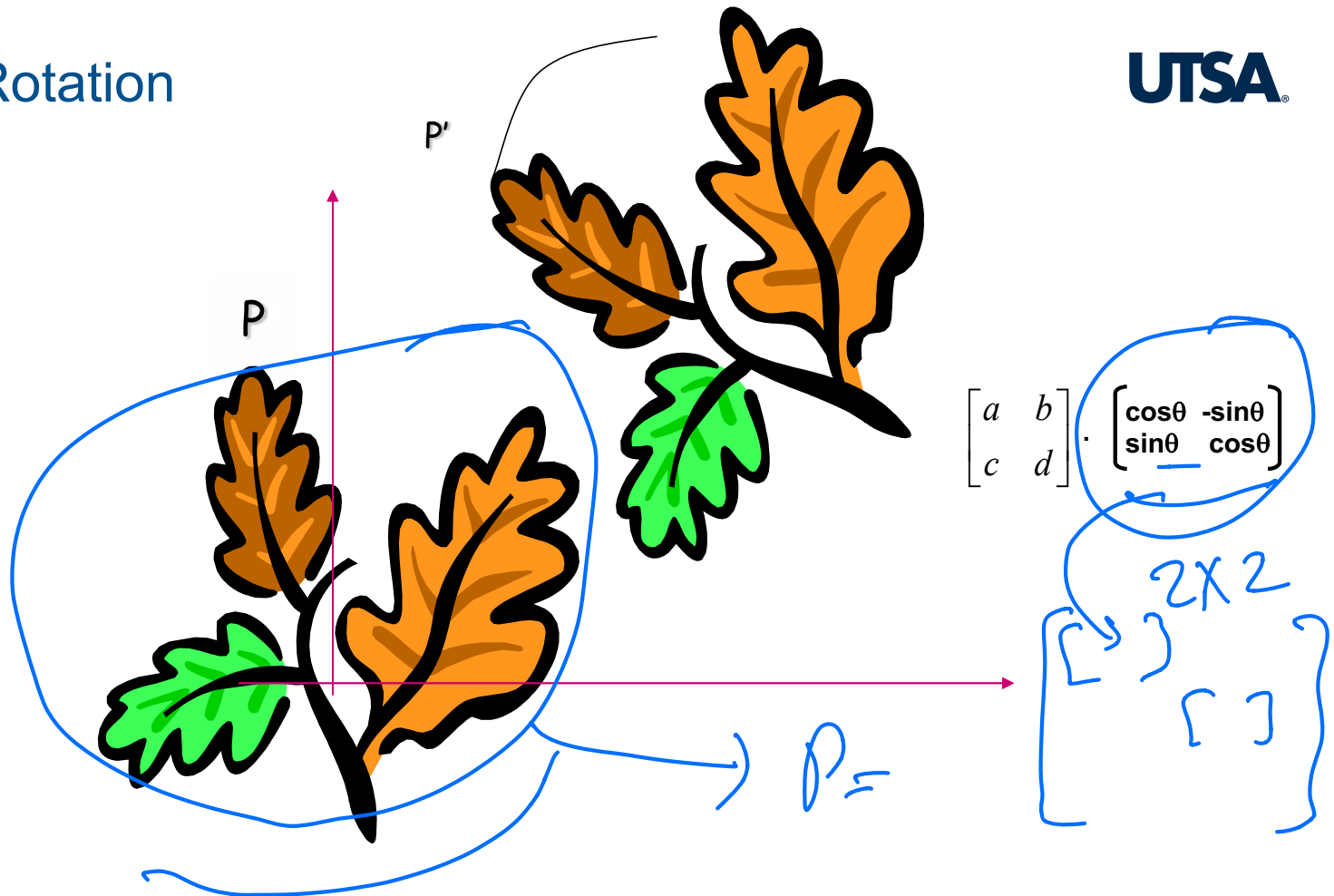
$$P \cdot \begin{bmatrix} 1.5 & & & \\ & 1.5 & & \\ & & \ddots & \\ & & & 1.8 \end{bmatrix}$$

100×100

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

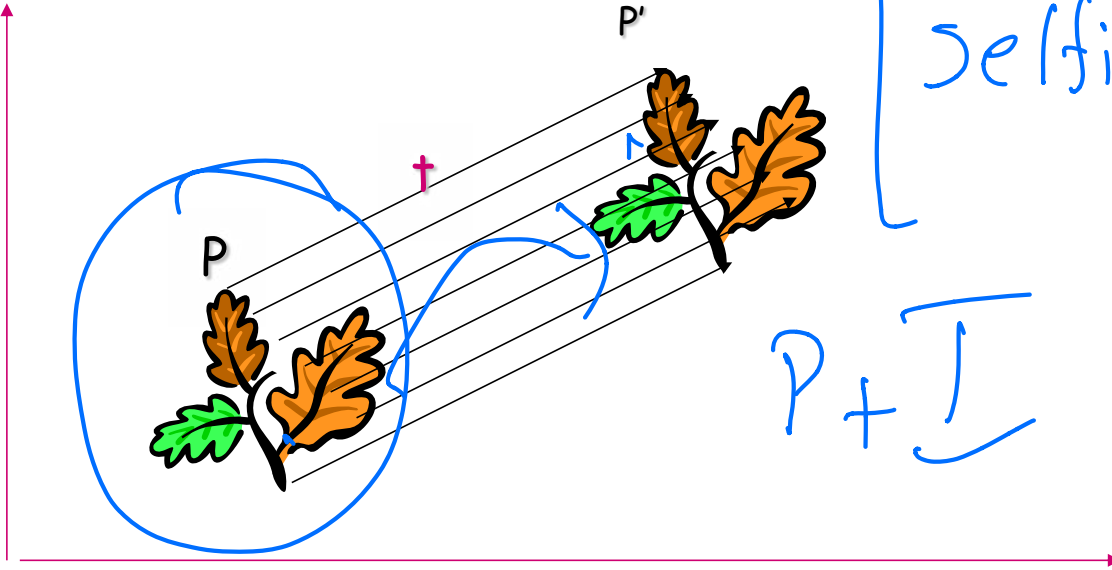
a.k.a: dilation ($r > 1$),
contraction ($r < 1$)

Rotation



2D Translation

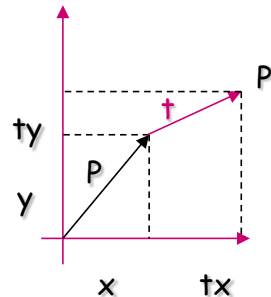
30fps



$$\begin{bmatrix} \text{selfie} \end{bmatrix} \times \underline{\underline{I}} = \underline{\underline{I}}$$

$$P + \underline{\underline{I}} = \underline{\underline{I}}$$

$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$



↓ Fps
120

Inverse of a Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{UTSA}$$

$I_1 = 1$ 3×3

- Identity matrix:

$$\mathbf{AI} = \mathbf{A}$$

- Inverse exists only for square matrices that are non-singular
 - Maps N-d space to another N-d space bijectively

- Some matrices have an inverse, such that:
 $\mathbf{AA}^{-1} = \mathbf{I}$

- Inversion is tricky:
 $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

Derived from non-commutativity property

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I_n : identity matrix of size $n \times n$

Determinant of a Matrix

- Used for inversion
- If $\det(A) = 0$, then A has no inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(A) =$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

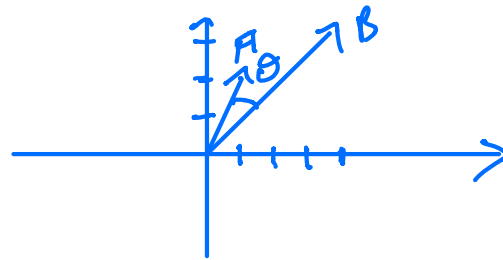
$$A^{-1} = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

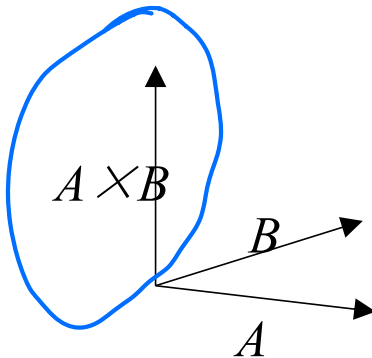
$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -\frac{1}{2} \end{bmatrix}$$

<http://www.euclideanspace.com/maths/algebra/matrix/functions/inverse/threeD/index.htm>

Vectors: cross product



- The cross product of vectors A and B is a vector C which is perpendicular to A and B
- The magnitude of C is proportional to the sin of the angle between A and B
- The direction of C follows the **right hand rule** if we are working in a right-handed coordinate system

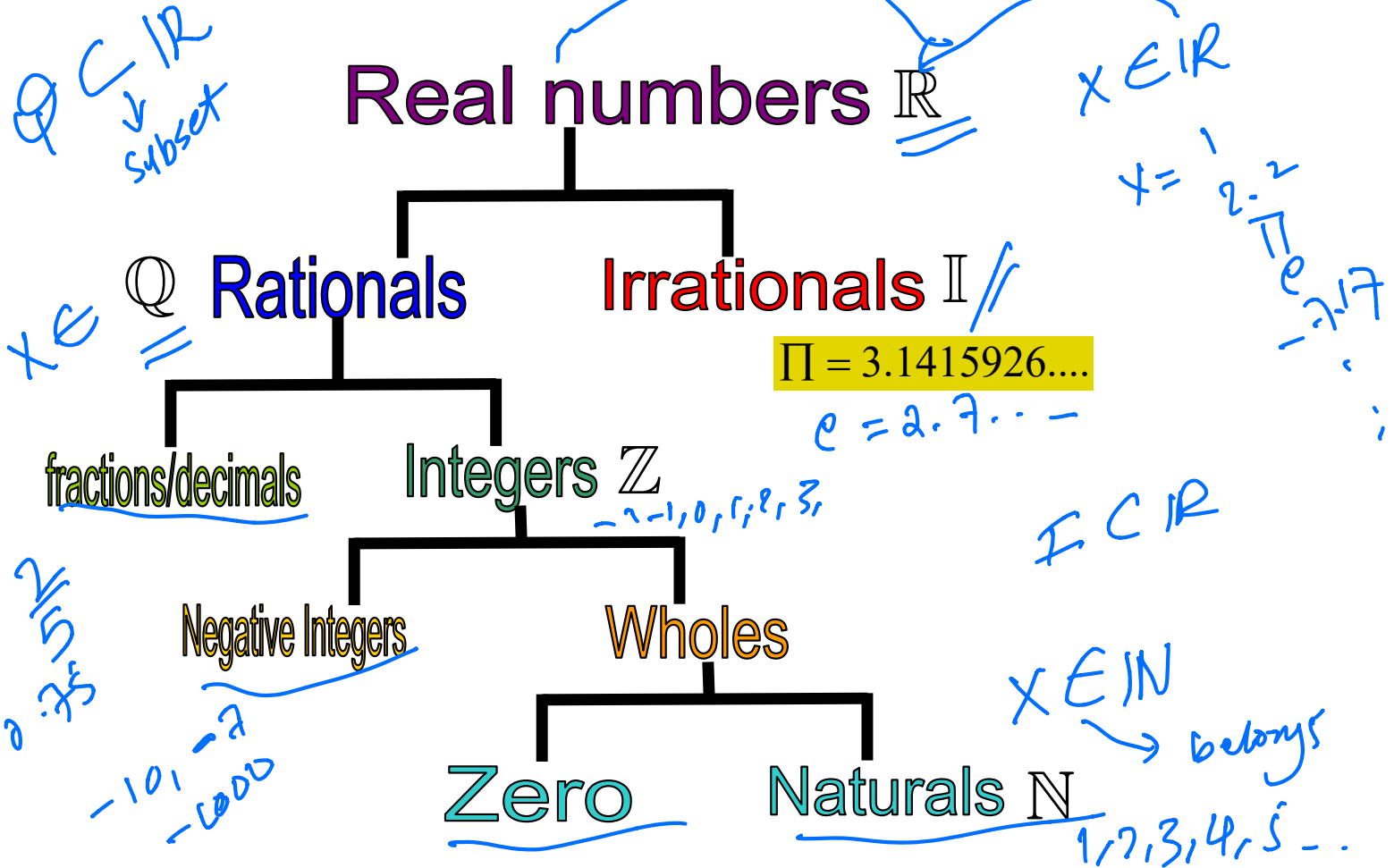


$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

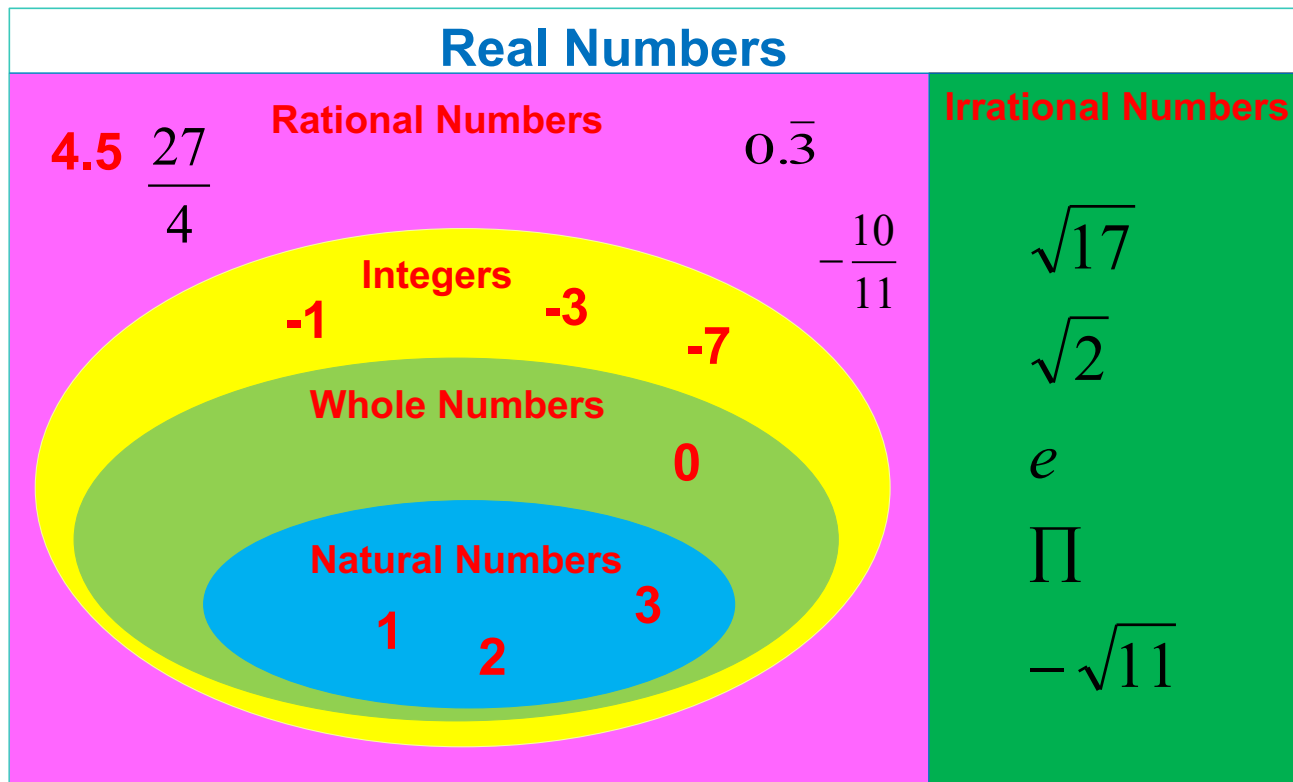
$$\|A \times B\| = \|A\| \|B\| \sin(\theta)$$

$$\|A \times B\| = \left(\sqrt{1^2 + 2^2} \right) \left(\sqrt{4^2 + 3^2} \right) \cdot \sin \theta$$

Numbers notation



Examples



Revision

$$\underline{x} \in \mathbb{R}, \quad \underline{\mathbf{x}} \in \mathbb{R}^n, \quad \underline{\mathbf{X}} \in \mathbb{R}^{n \times m}$$

- We learned a lot about scalars, vectors, and matrices
- Scalar: $x = 1$
- Vector (two notations):

$$\checkmark \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{x} = [1 \quad 1 \quad 4]^T$$

- Matrix:

$$\mathbf{X} = \begin{bmatrix} 1 & -1.1 & 4 \\ 2 & 10 & \pi \end{bmatrix}$$

$$\underline{\mathbf{X}} \in \underline{\mathbb{R}^{2 \times 3}}$$

$x \in \mathbb{R}^n$
x is a
vector
of dimension
n

$$x \in \mathbb{R}^n$$
$$x = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$x \in \mathbb{R}^3$$

Part 2:

Complex Numbers

Complex numbers: be real

$\mathbb{R}$: real #s
 $\mathbb{C}$: complex



- Math is used to explain our universe
- When a recurring phenomenon is seen and can't be explained by our present mathematics, new systems of mathematics are derived
- In the real number system, we can't take the square root of negative numbers, right?
- Therefore the complex number system was created
- Complex numbers exist in real-life and they are used in circuits to define voltages and currents
- Complex numbers revolutionized computer graphics, energy systems, imagining, video games, and so much more
- Similar to real numbers, complex scalars, vectors, and matrices exist:

$$\underline{x \in \mathbb{C}}, \quad \underline{\mathbf{x} \in \mathbb{C}^n}, \quad \underline{\mathbf{X} \in \mathbb{C}^{n \times m}}$$

$\mathbb{R}$
 $\mathbb{C}$

Where do complex numbers show up?

Solve both of these

$x \in \mathbb{C}$

$$x^2 - 81 = 0$$

$$x^2 = 81$$

$$x = \pm 9$$

$$x^2 + 81 = 0$$

$$x^2 = -81$$

Uhoh.....what do I do here?

Um, no solution?

- Does not have a real answer! $X = \pm\sqrt{-81}$ ✓
- It has an “imaginary” answer
- To define a complex number we have to create a new variable
- This new variable is “i”

New definition:

$$\boxed{i = \sqrt{-1}} \rightarrow i^2 = (\sqrt{-1})^2 = -1$$
$$= (-1)^{\frac{1}{2}}^2 = (-1)^1 = -1$$

- Note: i is the representation for $\sqrt{-1}$, **not** a simplification of $\sqrt{-1}$
- So, following this definition:

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1 = i^2 \cdot i^2$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$$

- Important note:

i = electric current

$$\sqrt{-1} \equiv i$$

$$\sqrt{-1} \equiv j$$

Math, Physics

Electrical Engineering

$2 + 5i$
Complex

And it cycles....

$$\checkmark i = \sqrt{-1}$$

$$i^5 = i^4 \cdot i = i$$

$$i^9 = i^8 \cdot i = i$$

$$\checkmark i^2 = -1$$

$$i^6 = i^4 \cdot i^2 = -1$$

$$i^{10} = i^8 \cdot i^2 = -1$$

$$\checkmark i^3 = -i$$

$$i^7 = i^4 \cdot i^3 = -i$$

$$i^{11} = i^8 \cdot i^3 = -i$$

$$\checkmark i^4 = 1$$

$$i^8 = i^4 \cdot i^4 = 1$$

$$i^{12} = i^8 \cdot i^4 = 1$$

Do you see a pattern yet?

$$i^{23}$$

$$i^{976} = ??$$

$$i^{24} =$$

Hints to deal with i

1. Find all “i”s at the beginning of a problem.
2. Treat all “i”s like variables, with all rules of exponents holding.
3. Reduce the power of i at the end by the rules we just learned..

$$2i + 2i = 23i$$

Examples

$$\underline{\underline{2i}} \quad , \quad \underline{\underline{2i}}$$

$$\sqrt{-4} = \sqrt{(-1)4} = \sqrt{-1} \cdot \sqrt{4} = i \cdot 2 = \underline{\underline{2i}}$$

$$\sqrt{-4} = \sqrt{-1} \sqrt{4}$$

$$\sqrt{-4} = j2 = \underline{\underline{2j}} = \underline{\underline{2i}}$$

$$1. \sqrt{-36} \cdot \sqrt{-81}$$

$$\sqrt{-1,36} \cdot \sqrt{-1,81} = \sqrt{36 \cdot i} \cdot \sqrt{81 \cdot i}$$

$$2. \sqrt{-36} + \sqrt{-81}$$

//

;

What is a complex number?

A complex number has two parts – a real part and an imaginary part

A complex number comes in the form

$$\overline{a} + bi$$

real imaginary

Handwritten examples and labels:

$$2 + 3i$$

↓ real ↓ imaginary

$$4i$$

↪

All real numbers are complex $\rightarrow 3 = 3 + 0i$

All imaginary numbers are complex $\rightarrow$

$$7i = 0 + 7i$$

Handwritten: $2 \rightarrow -$

Handwritten: $5 + 5i$

Again, treat the i as a variable and you will have no problems

When adding complex numbers,
add the real parts together and
add the imaginary parts together.

The diagram illustrates the addition of two complex numbers, $(3 + 7i)$ and $(8 + 11i)$. Red arrows point from the text labels to the corresponding parts of the expression: one arrow points from "imaginary part" to the $7i$ term, and another points from "imaginary part" to the $11i$ term. A third arrow points from "real part" to the 3 term, and a fourth points from "real part" to the 8 term. Blue circles are drawn around the $7i$ and $11i$ terms, and around the 3 and 8 terms. Below the expression, the result $11 + 18i$ is shown in a black box, with blue lines underlining the 11 and $18i$ terms.

$$(3 + 7i) + (8 + 11i)$$
$$11 + 18i$$

When subtracting complex numbers, be sure to distribute the subtraction sign; then add like parts.

$$\begin{aligned} & (5 + 10i) - (15 - 2i) \\ & 5 + 10i - 15 + 2i \\ & \boxed{-10 + 12i} \end{aligned}$$

When multiplying complex numbers,
use the distributive property and simplify.

$$\begin{aligned} (-8i)(7i) &= -56i^2 \\ &= 56 \end{aligned}$$

$$(3 - 8i)(5 + 7i)$$

$$\underline{15} + \underline{21i} - \underline{40i} - 56i^2$$

$$15 - 19i + \underline{56}$$

$$\boxed{71 - 19i}$$

Remember,
 $i^2 = -1$

To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the complex number in the denominator of the fraction.

* complex conjugate of a complex ~~#~~
 $a + bi$ is equal to $a - bi$

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2$$

The complex conjugate of $3 - 5i$ is $3 + 5i$.

$$\left(\frac{7 + 2i}{3 - 5i} \right) \cdot \frac{3 + 5i}{3 + 5i}$$

$$\frac{5}{2} = 2.5$$

$$\frac{(7 + 2i)(3 + 5i)}{(3 - 5i)(3 + 5i)} = 1 = \text{Real \#}$$

$$\frac{21 + 35i + 6i + 10i^2}{9 + 15i - 15i - 25i^2}$$

$$\frac{21 + 41i - 10}{9 + 25}$$

$$\frac{11 + 41i}{34}$$

$$\frac{11 + 41i}{34}$$

$$= \frac{11}{34} + \frac{41}{34}i$$

Try these

1. $(3 + 5i) - (11 - 9i)$ ✓

2. $(5 - 6i)(2 + 7i)$ ✓

3. $\frac{2 - 3i}{5 + 8i}$ ✓

4. $(19 - i) + (4 + 15i)$ ✓

Try these

$$1. (3 + 5i) - (11 - 9i) \quad -8 + 14i$$

$$2. (5 - 6i)(2 + 7i) \quad 52 + 23i$$

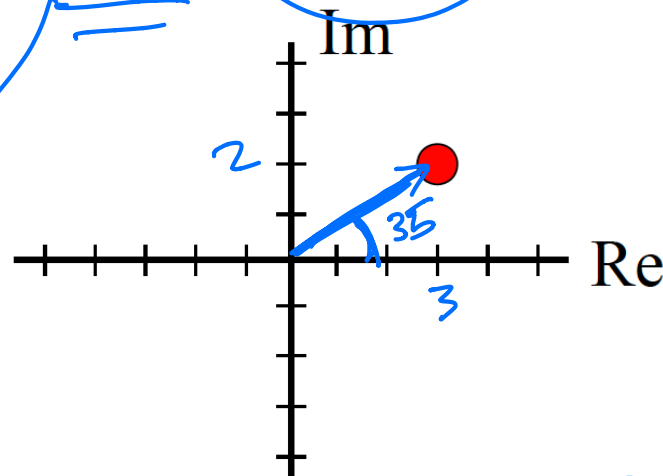
$$3. \frac{2 - 3i}{5 + 8i} \quad \frac{-14 - 31i}{89}$$

$$4. (19 - i) + (4 + 15i) \quad 23 + 14i$$

Graphing complex numbers

(rectangular coordinates)

Example: $3 + 2j \Rightarrow (3, 2)$



$$3 + 2i$$

$$= \sqrt{13} \angle 35^\circ$$

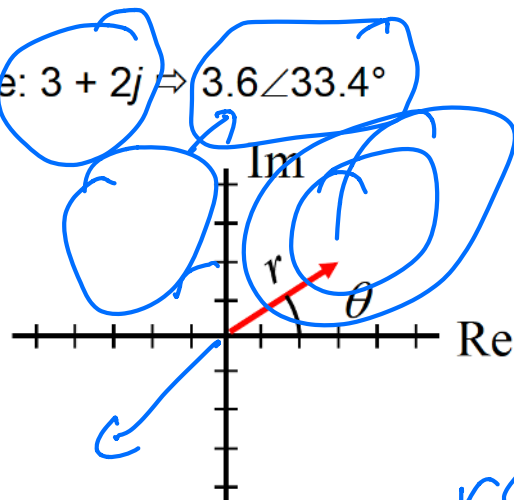
rectangular
rep

$$r \angle \theta$$

$$\left| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

Graphing complex numbers: polar form

Example: $3 + 2j \Rightarrow 3.6 \angle 33.4^\circ$



$$\text{Re} + j \text{Im} \Leftrightarrow r \angle \theta$$

$$\underline{\underline{r \angle \theta}}$$

$$r = \sqrt{\text{Re}^2 + \text{Im}^2} \quad \text{Re} = r \cos \theta$$

$$\theta = \tan^{-1} \left(\frac{\text{Im}}{\text{Re}} \right) \quad \text{Im} = r \sin \theta$$

$= \arctan \left(\frac{\text{Imag}}{\text{Real}} \right)$

Trigonometric form:

- Start with $\text{Re} + j \text{Im}$

$$2 + 3j$$

- Substitute

$$\text{Re} = r \cos \theta$$

$$\text{Im} = r \sin \theta$$

$$r \cos \theta + j r \sin \theta$$

$$r (\cos \theta + j \sin \theta)$$

$$\text{Re} + j \text{Im} = r (\cos \theta + j \sin \theta)$$

$$\underline{\text{magnitude}} \angle \underline{\text{angle}}$$

Euler's formula: $e^{j\theta} = \cos \theta + j \sin \theta$

- One of the most elegant equations of all time:

$$\boxed{e^{j\pi} + 1 = 0} \leadsto e^{j\pi} = -1$$

- Remember:

$r(\cos \theta + j \sin \theta)$ is equal to $r e^{j\theta}$

- Multiplication in polar:

$$\begin{aligned} & \underline{(r_1 \angle \theta_1) (r_2 \angle \theta_2)} \\ &= r_1 e^{j\theta_1} r_2 e^{j\theta_2} \end{aligned}$$

$$= r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$= r_1 r_2 \angle (\theta_1 + \theta_2)$$

$$\begin{aligned} & \underline{(a + bj)(c + dj)} \\ & \downarrow \\ & (r_1 \angle \theta_1) / (r_2 \angle \theta_2) \\ & = r_1 r_2 \angle \theta_1 + \theta_2 \\ & \rightarrow (a' + b'j) \\ & = r_1 / r_2 \angle \theta_1 - \theta_2 \end{aligned}$$

- Division in polar:

$$\begin{aligned} & \underline{(r_1 \angle \theta_1) / (r_2 \angle \theta_2)} \\ &= r_1 e^{j\theta_1} / r_2 e^{j\theta_2} \end{aligned}$$

$$= r_1 / r_2 e^{j(\theta_1 - \theta_2)}$$

$$= r_1 / r_2 \angle (\theta_1 - \theta_2)$$

$$\frac{a + bj \cdot c + dj}{c + dj \cdot c - dj}$$

Part 3:

Calculus: Limits, Derivatives, and Integrals

Limits

$$x \rightarrow c$$

$$\lim_{t \rightarrow \infty} \begin{matrix} i(t) \\ r(t) \end{matrix}$$

- Limits are fundamental in the world:
 - They help us understand how things evolve—and how to predict various phenomena

XEID

Definition of Limit

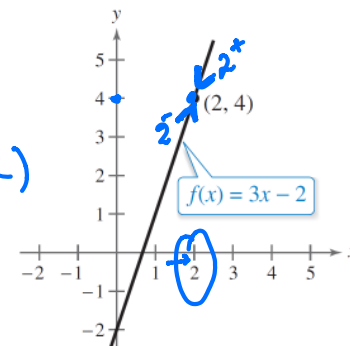
If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L . This is written as

$$\lim_{x \rightarrow c} f(x) = L$$

$$c=2, x \rightarrow 1.99 \rightarrow 2.01$$

- Example: estimate numerically the limit: $\lim_{x \rightarrow 2} (3x - 2)$ via the table

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300



$$f(x) = x^2 + 2x + 7; \quad x \cos(x); \quad e^{\tan(x)}; \quad \frac{1}{|x|}$$

Not all limits exist

find $f(x)$ if $x=2$

- Show that the limit of this function below does not exist

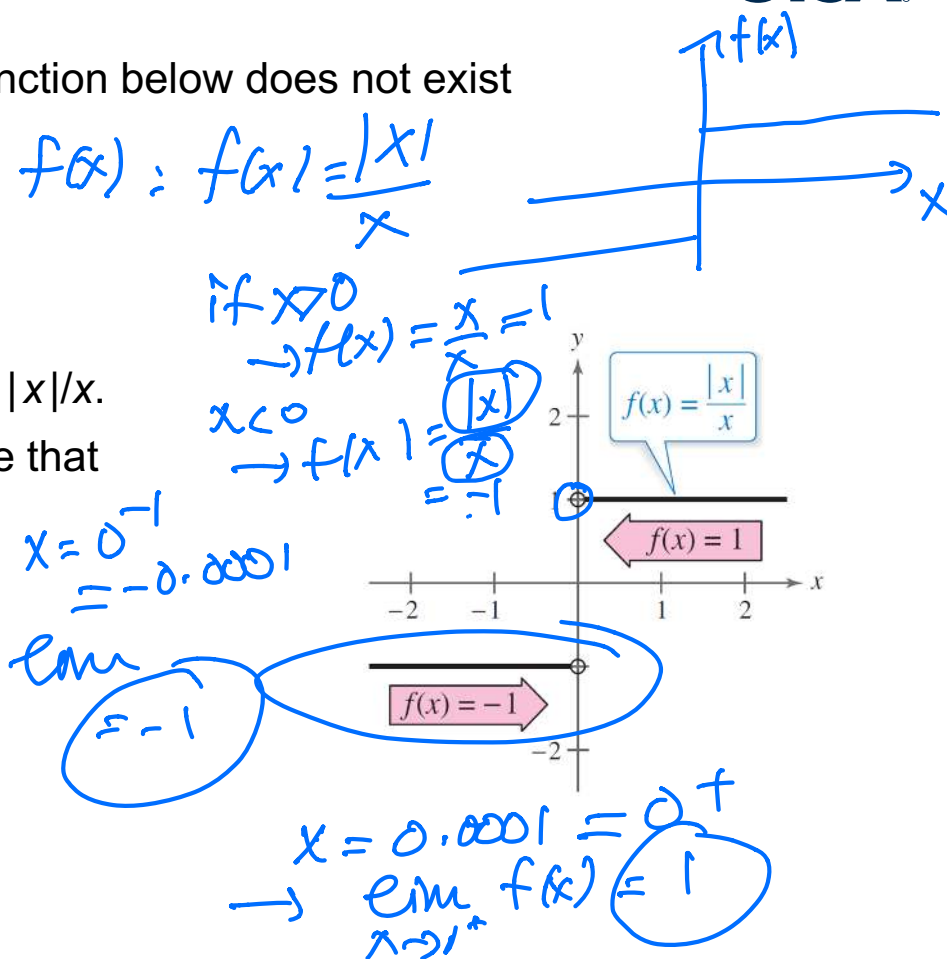
$$\lim_{x \rightarrow 0} \frac{|x|}{x} = \lim_{x \rightarrow 0} f(x) : f(x) = \frac{|x|}{x}$$

- Consider the graph of $f(x) = |x|/x$.
- From the figure, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

- and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0$$



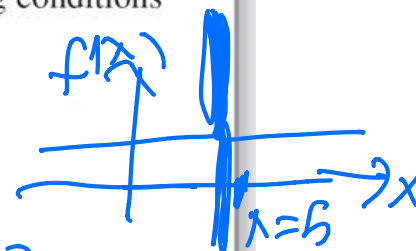
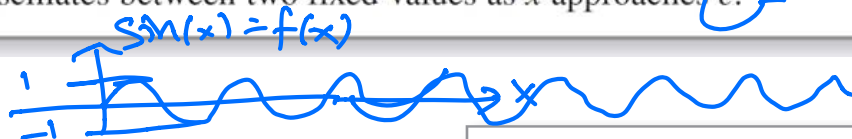
Properties of limits

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \rightarrow c$ does not exist when any of the following conditions are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c . (c^-)
2. $f(x)$ increases or decreases without bound as x approaches c .
3. $f(x)$ oscillates between two fixed values as x approaches c . $c = \infty$

c^+



Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$ $f(x) = b$ Limit of a constant function
2. $\lim_{x \rightarrow c} x = c$ $f(x) = x$ Limit of the identity function
3. $\lim_{x \rightarrow c} x^n = c^n$ Limit of a power function
4. $\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$ Limit of a radical function

Properties of Limits

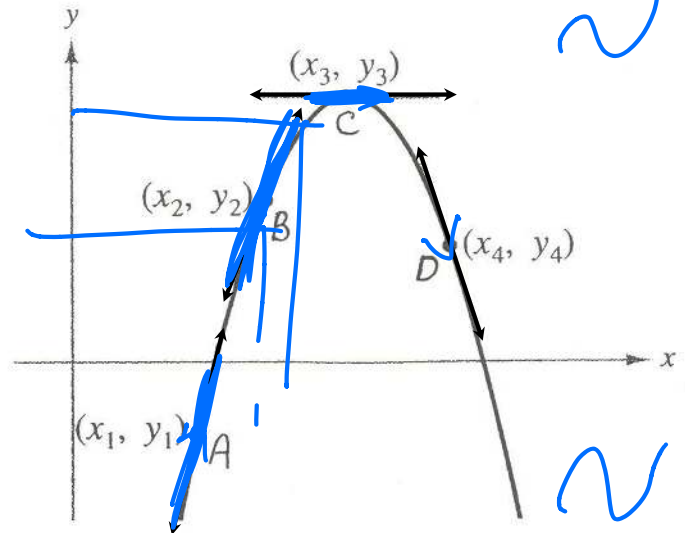
Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

1. Scalar multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, $K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

lim $f(x)$
lim $2f(x)$
 $= 2 \cdot \text{limit}$

Slopes and derivatives

- **Slope:** the rate at which a graph (function, line, plot) rises or falls
- For a straight line, the rate (or slope) is the same at every point on the line
- For graphs other than lines, the rate at which the graph rises or falls changes from point to point
- This parabola is rising more quickly at point A than it is at point B
- At the vertex, point C, the graph levels off
- At point D the graph is falling
- Find the slope of the tangent line to the point
→ **rate of change**
- How do we calculate the slope of a tangent line (i.e., rate of change or derivative)?



Derivatives and slopes

$$f'(x) = \text{derivative} = \text{slope}$$

- The derivative is the slope of the tangent line to a graph $f(x)$, and is usually denoted $f'(x)$
- To calculate the slope of the tangent line we will use the difference quotient:

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

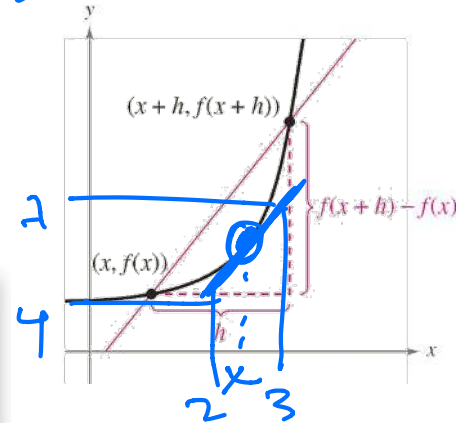
$$m = \lim_{h \rightarrow 0} m_{\text{sec}}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

theory

$$f(x) = \sin(x) \\ \rightarrow f'(x) = \cos(x)$$



$$\frac{7-4}{3-2} = \frac{3}{1}$$

slope = fixed value at one point
derivative = varying slope

- Example: Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$
- Solution:** Find an expression that represents the slope of a secant line at $(-2, 4)$

Example solution

$$\underline{x = -2} \quad f(x) = (-2)^2 = 4$$

- Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$

$$\begin{aligned} f(x) &= x^2 \\ f(x+h) &= (x+h)^2 \\ &= (x+h)^2 \\ &= (h-2)^2 \\ &= (-2+h)^2 \end{aligned}$$

$$m_{\text{sec}} = \frac{f(-2+h) - f(-2)}{h}$$

Set up difference quotient.

$$= \frac{(-2+h)^2 - (-2)^2}{h}$$

Substitute in $f(x) = x^2$.

$$= \frac{4 - 4h + h^2 - 4}{h}$$

Expand terms.

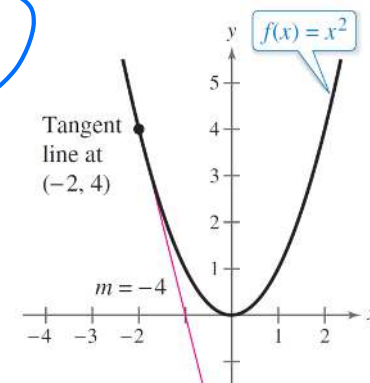
$$= \frac{-4h + h^2}{h} = \frac{\cancel{h}(-4 + h)}{\cancel{h}} = -4 + h, \quad h \neq 0$$

Simplify.

- Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \rightarrow 0} m_{\text{sec}} = \lim_{h \rightarrow 0} (-4 + h) = -4$$

$$\begin{aligned} f(x) &= x^2 \rightarrow f'(x) = 2x \\ &= 2(-2) \\ &= -4 \end{aligned}$$



Derivatives

- Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$

Definition of Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

$f'(x)$

Find the derivative of $f(x) = 3x^2 - 2x$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 - 2h}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{h(6x + 3h - 2)}{h}$$

$$= \lim_{h \rightarrow 0} (6x + 3h - 2)$$

$$= 6x + 3(0) - 2$$

$$= 6x - 2$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2.$$

Computing derivatives immediately

- The aforementioned method is sorta tedious
- Need closed-form expressions. And we have them

$$f(x) = \ln(x) \rightarrow f'(x) = \frac{1}{x}$$

$$f(x) = x^4 \rightarrow f'(x) = 4x^3$$

$$f(x) = 3x^4 + 2x^2 \rightarrow f'(x) = 3(4x^3) + 2(2x) = 12x^3 + 4x$$

$$f(x) = 7 \rightarrow f'(x) = 0$$

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	e^x	e^x
$\ln(x)$	$1/x$	$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$	$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\operatorname{cosec}^2(x)$	$\sec(x)$	$\sec(x)\tan(x)$
$\operatorname{cosec}(x)$	$-\operatorname{cosec}(x)\cot(x)$	$\tan^{-1}(x)$	$1/(1+x^2)$
$\sin^{-1}(x)$	$1/\sqrt{1-x^2}$ for $ x < 1$	$\cos^{-1}(x)$	$-1/\sqrt{1-x^2}$ for $ x < 1$
$\sinh(x)$	$\cosh(x)$	$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\operatorname{sech}^2(x)$	$\coth(x)$	$-\operatorname{cosech}^2(x)$
$\operatorname{sech}(x)$	$-\operatorname{sech}(x)\tanh(x)$	$\operatorname{cosech}(x)$	$-\operatorname{cosech}(x)\coth(x)$
$\sinh^{-1}(x)$	$1/\sqrt{x^2+1}$	$\cosh^{-1}(x)$	$1/\sqrt{x^2-1}$ for $x > 1$
$\tanh^{-1}(x)$	$1/(1-x^2)$ for $ x < 1$	$\coth^{-1}(x)$	$-1/(x^2-1)$ for $ x > 1$

$$d/dx$$

Anti-derivatives and indefinite integrals

- Suppose we have a graph of the derivative of a function $f'(x)$
- What do we know about the original function?
 - Critical numbers
 - Where it is increasing, decreasing
- Can we recover the original function?
- **Derivatives** give us the rate of change of a function
- What if we know the rate of change: Can we find the original function?
- If $F'(x) = f(x)$
 - Then **$F(x)$ is an antiderivative of $f(x)$**
- Example:
 - Let $F(x) = 12x^2$
 - Then $F'(x) = 24x = f(x)$
 - So $F(x) = 12x^2$ is the antiderivative of $f(x) = 24x$
- Building on that, the integration $\int f(x)dx$ computes the antiderivative of a function $f(x)$
- More on that in your calculus courses ☺

$f(x) \rightarrow f'(x)$
↓
rate of change

✓
 $\int f'(x) \rightarrow \underline{\underline{f(x)}}$

$$(x^n)' = nx^{n-1}$$

$\int \int$ integration symbol



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EE 1322: Intro to Electrical & Computer Engineering

Module 3: Math Basics and Background

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