Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 5 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.
- Question 6 is a bonus question. You do not need to answer it.

Question Number	Maximum Points	Your Score
1	20	
2	20	
3	25	
4	20	
5	15	
Total	100	

1. (20 total points) Find the Laplace transform or the inverse Laplace transform for the following functions. You may use the LT table.

   (a) (5 points) $f_1(t) = 3e^{2t}\cos(11t) + 2e^{-3t}\sinh(7t)$.

   $$\boxed{F_1(s) = \frac{3s-6}{(s-2)^2+121} + \frac{14}{(s+3)^2-49}.}$$

   (b) (5 points) $F_2(s) = \dfrac{\sqrt{135}}{(s^2+3)^2}$.

   $$\boxed{f_2(t) = \frac{\sqrt{5}}{2}\sin(\sqrt{3}t) - \frac{\sqrt{15}}{2}t\cos(\sqrt{3}t).}$$

   (c) (5 points) $f_3(t) = e^{2t}(t^3 + 5t - 2\cos(\frac{\pi t}{4}))$.

   $$\boxed{F_3(s) = \frac{5}{(s-2)^2} - \frac{2s-4}{(s-2)^2+\left(\frac{\pi}{4}\right)^2} + \frac{6}{(s-2)^4}.}$$

   (d) (5 points) $F_4(s) = \dfrac{3s+5}{s(s^2+2s+5)}$.

   $$\boxed{f_4(t) = 1 - e^{-t}\cos(2t) + e^{-t}\sin(2t).}$$

2. (20 total points) The following ODE is given:

$$y''(t) + 15y'(t) + 56y(t) = u(t)$$

   (a) (20 points) Given that $y(0) = 0$ and $y'(0) = 0$, first compute $H(s) = \frac{Y(s)}{U(s)}$ then $y(t)$ if $u(t) = \pi \approx 3.14$.

   Taking the Laplace transform for the ODE, and considering the given initial conditions, we obtain:

$$s^2Y(s) + 15sY(s) + 56Y(s) = \frac{\pi}{s}$$

   or

$$Y(s) = \frac{\pi}{s(s^2 + 15s + 56)} = \frac{\pi}{s(s+7)(s+8)}.$$

   Applying partial fraction expansion, we obtain:

$$Y(s) = \frac{\pi}{s(s+7)(s+8)} = \frac{A}{s} + \frac{B}{s+7} + \frac{C}{(s+8)},$$

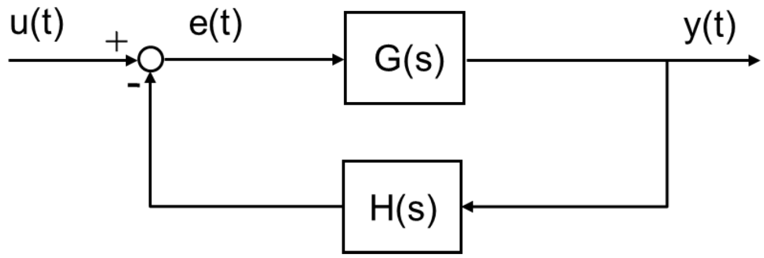
   where $A = \frac{\pi}{56}, B = -\frac{\pi}{7}$, and $C = \frac{\pi}{8}$.

   Therefore,

$$\boxed{y(t) = \frac{\pi}{56} - \frac{\pi}{7}e^{-7t} + \frac{\pi}{8}e^{-8t}.}$$

3. (25 total points) For the system given in the below figure, assume that:

$$G(s) = \frac{1}{(s-1)(s+2)}, \quad H(s) = s+3.$$



(a) (5 points) Find the transfer function $\frac{Y(s)}{U(s)}$ in the most simplified form.

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{(s-1)(s+2)}}{1+\frac{s+3}{(s-1)(s+2)}} = \frac{1}{(s-1)(s+2)+s+3} = \frac{1}{s^2+2s+1}.$$

(b) (5 points) Find $Y(s)$ if $u(t) = 2$. DO NOT compute $y(t)$.

$$Y(s) = \frac{U(s)}{s^2+2s+1} = \frac{2}{s(s^2+2s+1)} = \frac{2}{s(s+1)^2}.$$

(c) (5 points) What are the poles of $Y(s)$? Does the final value of $y(t)$ exist (i.e., $y(\infty)$)? If it does, find it via the **final value theorem**. Otherwise, tell me why it doesn't.

The poles of $Y(s)$ are: $0, -1, -1$. Therefore, the final value theorem applies:

$$y(\infty) = lim_{s\to 0}\, sY(s) = 2.$$

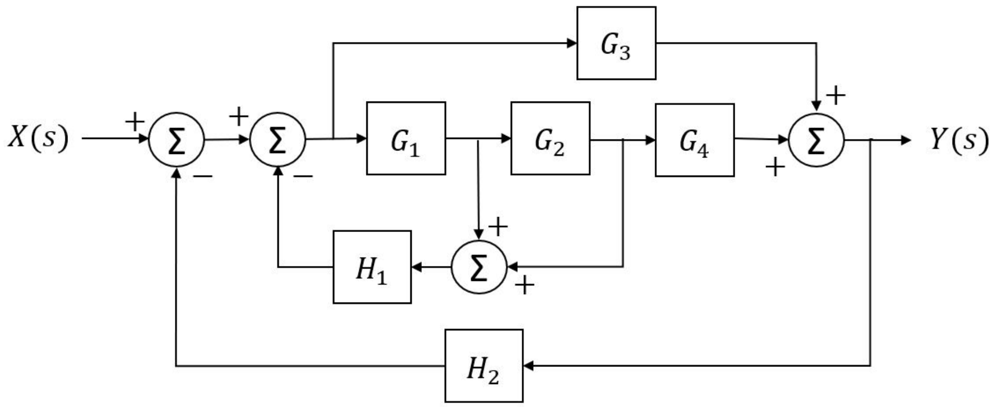
(d) (10 points) Obtain $\frac{E(s)}{U(s)}$, then find $E(s)$ for the given $u(t) = 2$. Does the final value of $e(t)$ exist (i.e., $e(\infty)$)? If it does, find it via the **final value theorem**. Otherwise, tell me why it doesn't.

$$\frac{E(s)}{U(s)} = \frac{1}{1+G(s)H(s)} \Rightarrow E(s) = \frac{(s-1)(s+2)}{(s-1)(s+2)+s+3}\cdot\frac{2}{s} = 2\frac{(s-1)(s+2)}{s(s+1)^2}.$$

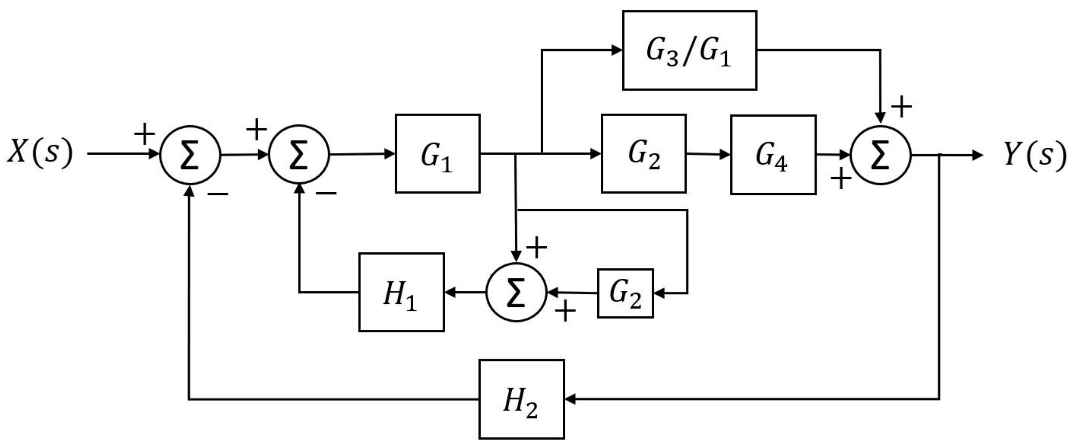
The poles of $E(s)$: $0, -1, -1$. Therefore, the final value theorem applies:

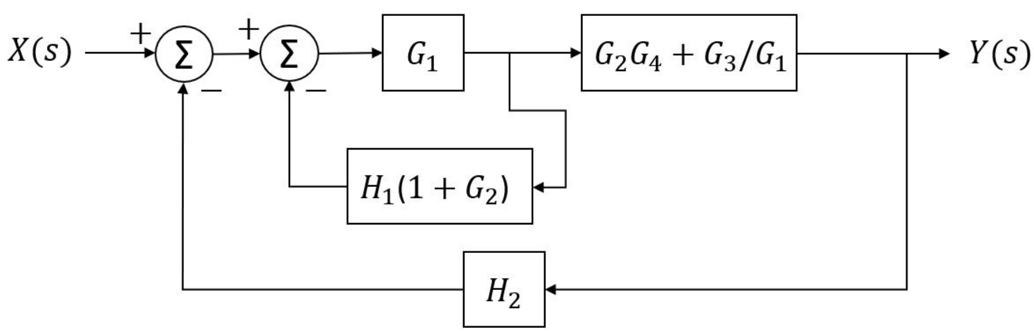
$$e(\infty) = lim_{s\to 0}\, sE(s) = -4.$$

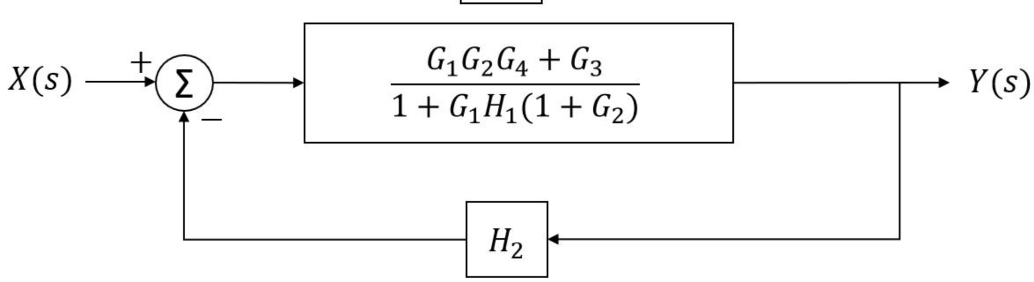
4. (20 total points) You are given the following block diagram.

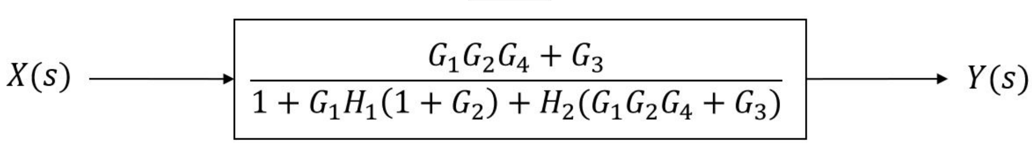


(a) (20 points) Find $\dfrac{Y(s)}{X(s)}$ for the above system. Show your work.

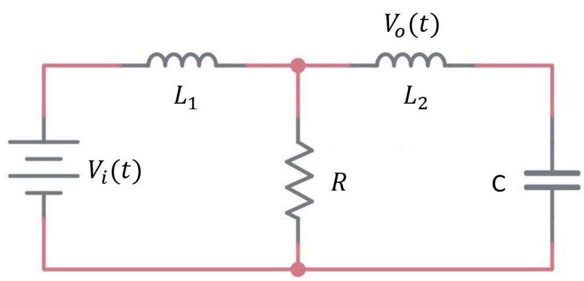








5. (15) You are given the following RLC circuit.



(a) (15 points) Derive the transfer function $\frac{V_o(s)}{V_i(s)}$ in terms of R, L_1, L_2 and C. Show your work. Your transfer function should have the standard form of a transfer function, i.e., polynomials in the numerator and denominator.

Notice that the voltages of an inductor and a capacitor in Laplace domain are, in general, given as

$$V(s) = sLI(s)$$

$$V(s) = \frac{1}{sC}I(s).$$

Hence using voltage division, we obtain the first equation

$$V_o(s) = \frac{sL_2}{sL_2 + \frac{1}{sC}} V_R(s).$$

where $V_R(s)$ is the voltage across resistor R. The second equation is given as

$$V_R(s) = \frac{R \parallel \left(sL_2 + \frac{1}{sC}\right)}{sL_1 + R \parallel \left(sL_2 + \frac{1}{sC}\right)} V_i(s),$$

where $\parallel$ denotes the parallel connection. From both equations, we obtain

$$\frac{V_o(s)}{V_i(s)} = \frac{sL_2}{sL_2 + \frac{1}{sC}} \cdot \frac{R \parallel \left(sL_2 + \frac{1}{sC}\right)}{sL_1 + R \parallel \left(sL_2 + \frac{1}{sC}\right)},$$

which can be simplified into

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{s^2RL_2}{s^3L_1L_2 + s^2R(L_1 + L_2) + s\frac{L_1}{C} + \frac{R}{C}}.}$$