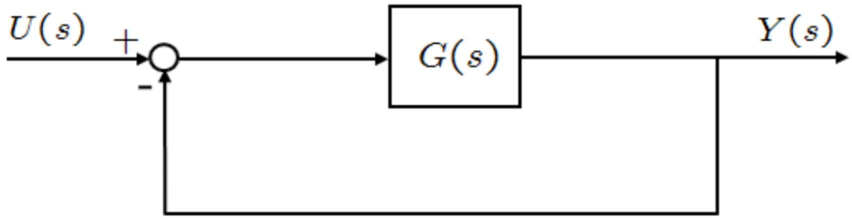
Your Name: | Your Signature:

- **Exam duration:** 1 hour and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place $\boxed{\text{a box around your final answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 11 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.
- **Question 6 is a bonus question. You do not need to answer it. You can, however, choose to answer it instead of another exam question. The maximum achievable grade is 110.**

Question Number	Maximum Points	Your Score
1	20	
2	20	
3	25	
4	15	
5	20	
Total	100	
Bonus	20	



1. (20 total points) For the system shown in the above figure, assume that $G(s) = \dfrac{1}{(s+2)^2}$.

(a) (2 points) What is the closed-loop transfer function (CLTF)?

$$G_{CLTF}(s) = \frac{G(s)}{1+G(s)} = \frac{1}{(s+2)^2+1} = \frac{1}{s^2+4s+5}.$$

(b) (2 points) What are the poles of the CLTF?

Poles are: $p_{1,2} = -2 \pm j$.

(c) (3 points) Find the settling time via the 5% criterion ($t_s = \dfrac{3}{\zeta\omega_n}$).

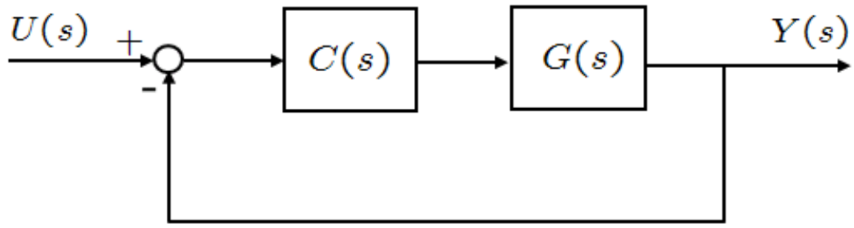
$\omega_n = \sqrt{5}, \zeta = 2/\sqrt{5}$. Settling time is $t_s = \dfrac{3}{\zeta\omega_n} = \dfrac{3}{2} = 1.5\,\text{sec}$

(d) (3 points) Compute the steady-state error corresponding to a unit step input—you can (should) use the provided SSE table.

System is Type 0, then the SSE is $e_{ss} = \dfrac{1}{1+G(0)} = 0.8$.

(e) (10 points) We want to design the system such that $t_s = 1\,\text{sec}$ and damping ration $\zeta = \frac{\sqrt{3}}{2}$. Find the corresponding desired CLTF poles given these design objectives.

Given that the settling time is 1 second, we can find $\zeta_d\omega_{nd}$ since $t_s = \dfrac{3}{\zeta\omega_n}$. Hence, $\zeta_d\omega_{dn} = 3 = \sigma_d$. Also, we're given that the damping ratio is $\zeta_d = \frac{\sqrt{3}}{2} \Rightarrow \omega_{dn} = 6/\sqrt{3}$. Therefore, the desired closed loop poles are generated from the desired CLTF denominator $(s^2+6s+12)$, and they are: $p^d_{1,2} = -3 - \pm\sqrt{3}j$.



2. (20 total points) For the system shown in the above figure, assume that:

$$G(s) = \frac{1}{s^3 + s^2 + 2s - 0.5}\ ,\ C(s) = 1 + \frac{K}{s}\ ,\ K \geq 0.$$

(a) (5 points) Find the CLTF.

$$G_{CLTF}(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{s + K}{s^4 + s^3 + 2s^2 + 0.5s + K}$$

(b) (5 points) Using the Routh array criterion, find the range of K such that the system is stable.

$$\begin{array}{c||ccc} s^4 & 1 & 2 & K \\ s^3 & 1 & 0.5 & 0 \\ s^2 & 1.5 & K & \\ s^1 & \frac{0.75-K}{1.5} & & \\ s^0 & K & & \end{array}$$

Hence, the range of K s.t. the system is stable is $0 < K < 0.75$.

(c) (5 points) In terms of K, obtain the SSE of the above system given that the input is a unit ramp function. You'll have to figure out the System Type and use the SSE table (see last page of your exam booklet).

The system is of Type 1, hence the steady state error for this system given a ramp input is

$$e_{ss} = \frac{1}{\lim_{s\to 0} sC(s)G(s)} = -\frac{1}{2K}.$$

Note that this expression only holds if $0 < K < 0.75$.

(d) (5 points) By varying the value of K, what is the smallest absolute value of such tracking error one can achieve (this part is related to (c) above)?

We want $\frac{1}{2K}$ to be as small as possible. Hence, we want K to be as close to 0.75 as possible, but not 0.75 (since $K = 0.75$ would make the system marginally stable). Therefore, the smallest value would be $e_{ss}^{\min} = 1/(1.5) = 0.66$.

3. (25 total points) The characteristic polynomial (CP) of a system is given as follows:

$$1 + KH(s)G(s) = 1 + K\frac{1}{(s^2+2s+2)(s-1)}.$$

We want to sketch the root locus in this problem of the given CP.

(a) (5 points) Find the part of the real axis on the root locus. How many branches does the root locus have?

The poles are $p_{1,2,3} = -1 \pm j, +1$. The points on the real axis to the left of $p_3 = 1$ belong to the root locus.

(b) (5 points) Find the asymptotes of the root locus, including their angles (ϕ_q) and point of intersection (σ_a).

$n_p - n_z = 3$, hence we have 3 branches. The angles of asymptotes are: $\phi_q = 60, 180, -60\,\text{deg}$ and they all intersect at $\sigma_a = -0.33$.

(c) (5 points) Find the break-in/breakaway points of the root locus.

Setting the CP to zero, we obtain:

$$K(s) = -(s^3 + s^2 - 2) \rightarrow dK/ds = -(3s^2 + 2s) = 0 \Rightarrow s_{1,2} = -2/3, 0.$$

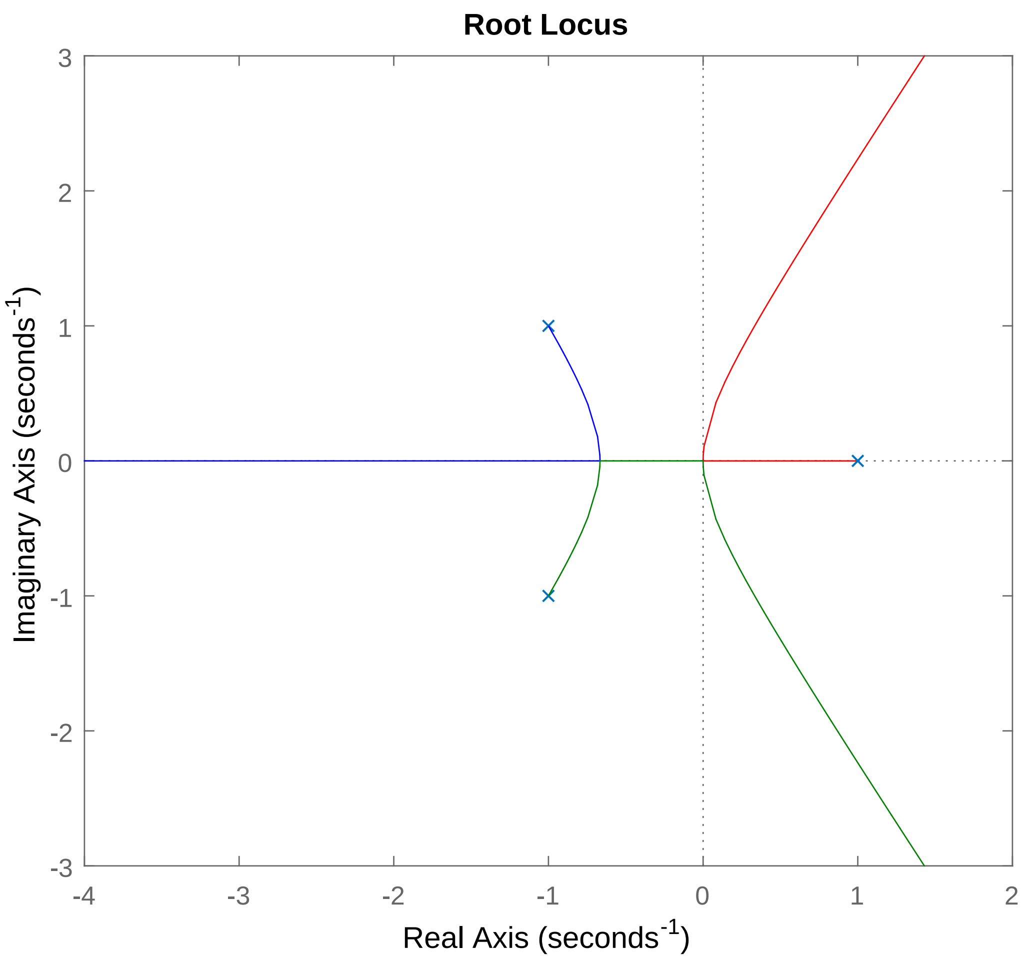
Both of these points belong to the root locus: s_1 is a break-in point and s_2 is a breakaway point.

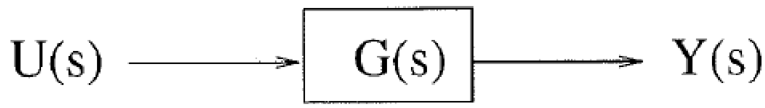
(d) (5 points) Find the angle of departure from the complex poles (you can only find one of these angles, since the RL is symmetric). You are also given that $\tan^{-1}(1/2) = 26.56\,\text{deg}$.

Angle of departure from p_1: $\phi_{p_1} = 180 - 90 + (-180 + 26.6) = -63.4\,\text{deg}$. Similarly, $\phi_{p_2} = +63.4\,\text{deg}$.

(e) (5 points) Find the crossings with the $j\omega$ axis, and sketch the root locus.

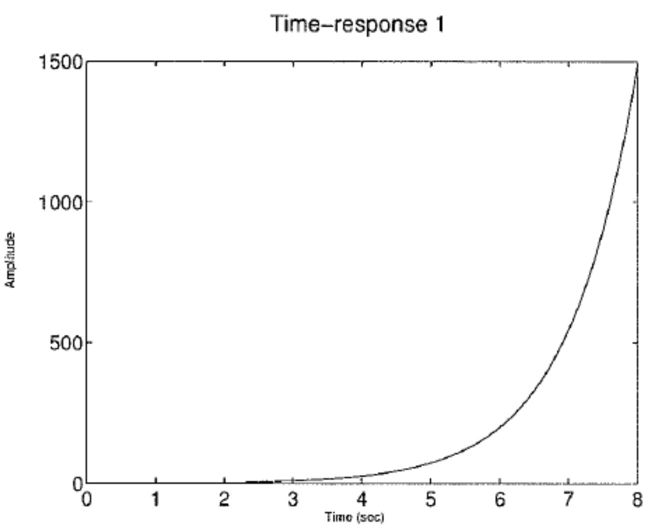
Crossings occur at $\omega = 0$ and $K = 2$. Root locus plot:

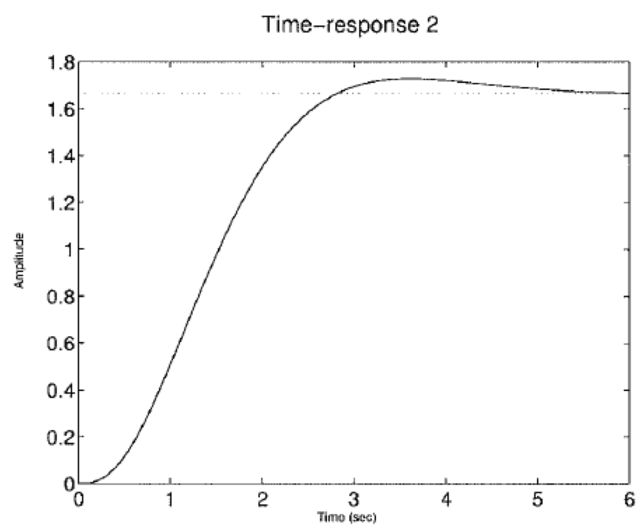

Root Locus
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2

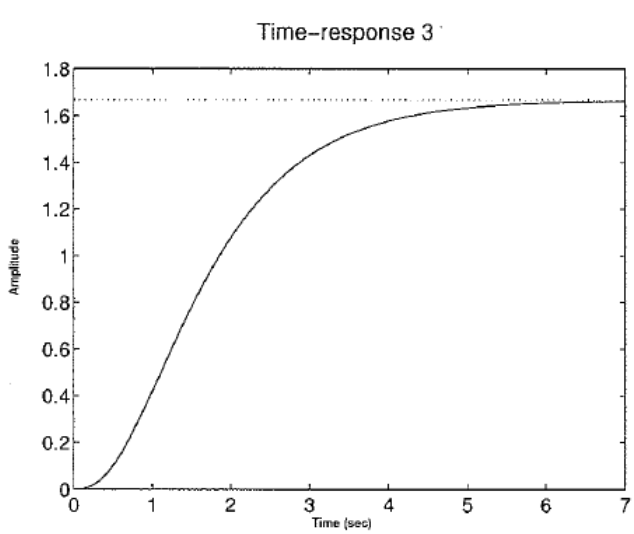


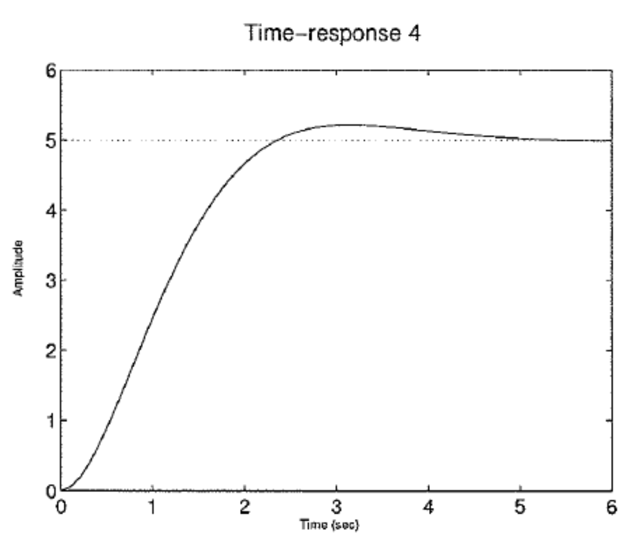
4. (15 total points) For the above system, match the four given step-response plots given in the below figure with the four transfer functions given in the below table. **Justify your answer in clear sentences. Lazy justifications get lazy credits.**

Transfer Function $G(s)$	$\frac{10}{(s+1)(s+2)(s+3)}$	$\frac{10}{(s^2+2s+2)(s+3)}$	$\frac{1}{(s-1)(s+1)}$	$\frac{10}{s^2+2s+2}$
Step Response Plot #	??	??	??	??









Your solution here:

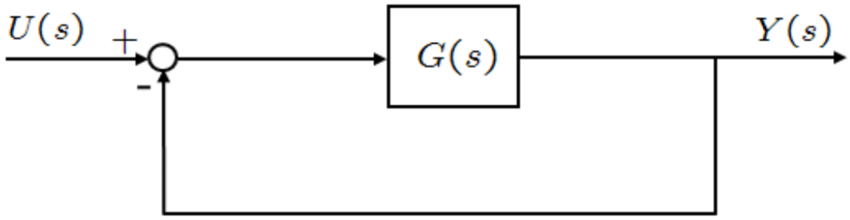
- $\dfrac{10}{(s+1)(s+2)(s+3)}$: This transfer function corresponds to **time response plot 3**, as the poles are **real and in the LHP**, which signifies no overshoot and the step response approaches the final value.
- $\dfrac{10}{(s^2+2s+2)(s+3)}$: This transfer function corresponds to **time response plot 2**, as the poles are **complex and in the LHP**, which signifies an overshoot and the step response approaches the final value. To determine the final value, we can use the final value theorem which gives

$$y(\infty) = \lim_{s\to 0} sU(s)G(s) = \lim_{s\to 0} G(s) = 10/6 = 1.6$$

  which is shown in time response 2.
- $\dfrac{1}{(s-1)(s+1)}$: This system is unstable with a pole $p_1 = 1$ in the RHP. Hence the step-response will be unstable, which is only shown in **time response plot 1**.
- $\dfrac{10}{s^2+2s+2}$: Similar to the solution of time-response 2, the solution for this transfer function corresponds to **time response plot 4:**

$$y(\infty) = \lim_{s\to 0} sU(s)G(s) = \lim_{s\to 0} G(s) = 10/2 = 5.$$

Transfer Function $G(s)$	$\dfrac{10}{(s+1)(s+2)(s+3)}$	$\dfrac{10}{(s^2+2s+2)(s+3)}$	$\dfrac{1}{(s-1)(s+1)}$	$\dfrac{10}{s^2+2s+2}$
Step Response Plot #	**3**	**2**	**1**	**4**



5. (20 total points) For the above system, consider that

$$G(s) = \frac{K}{(s+3)(s+1)},\ K \geq 0.$$

(a) (5 points) Find the closed loop transfer function.

$$G_{CLTF}(s) = \frac{K}{s^2+4s+K+3}.$$

(b) (5 points) What are ω_n and ζ for this standard second order system in terms of K?

$$\omega_n = \sqrt{K+3}, \zeta = \frac{2}{\sqrt{K+3}}.$$

(c) (10 points) Find the value of K is the maximum overshoot is equal to 0.1. The maximum overshoot is given by the following formula:

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi},$$

and $\dfrac{\ln(0.1)}{\pi} \approx -0.75$.
Your solution (K) can be approximate, but you have to clearly show all the steps involved.

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = 0.1 \Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} \approx 0.75 \Rightarrow \zeta \approx 0.6 \Rightarrow \frac{2}{\sqrt{K+3}} = 0.6 \Rightarrow K \approx 8.5.$$

6. (20 total points) The characteristic polynomial of a closed-loop system is given by:

$$(1+K)s^2 + (2-2K)s + 2K = 0.$$

(a) (20 points) Plot the root locus. You should follow **all the steps we discussed in class**. A simple sketch will not be graded. You are also given that $\tan^{-1}(1) = 45\,\text{deg}$ and $\tan^{-1}(1/3) = 18.43\,\text{deg}$

(0) Using the hint, we can write the given characteristic polynomial as a standard $1 + KG(s) = 0$:

$$(1+K)s^2 + (2-2K)s + 2K = 0 \Rightarrow s^2 + 2s + K(s^2 - 2s + 2) = 0$$

$$\Rightarrow 1 + K\frac{s^2 - 2s + 2}{s^2 + 2s} = 0.$$

Hence, the new $G(s)$, is equal to $\tilde{G}(s) = \dfrac{s^2 - 2s + 2}{s^2 + 2s}$.

1. Poles: $p_{1,2} = 0, -2$. Zeros: $z_{1,2} = 1 \pm j$. $n_p = 2, n_z = 2$
2. Asymptotes: None, since $n_p - n_z \leq 0$
3. Breakaway points:

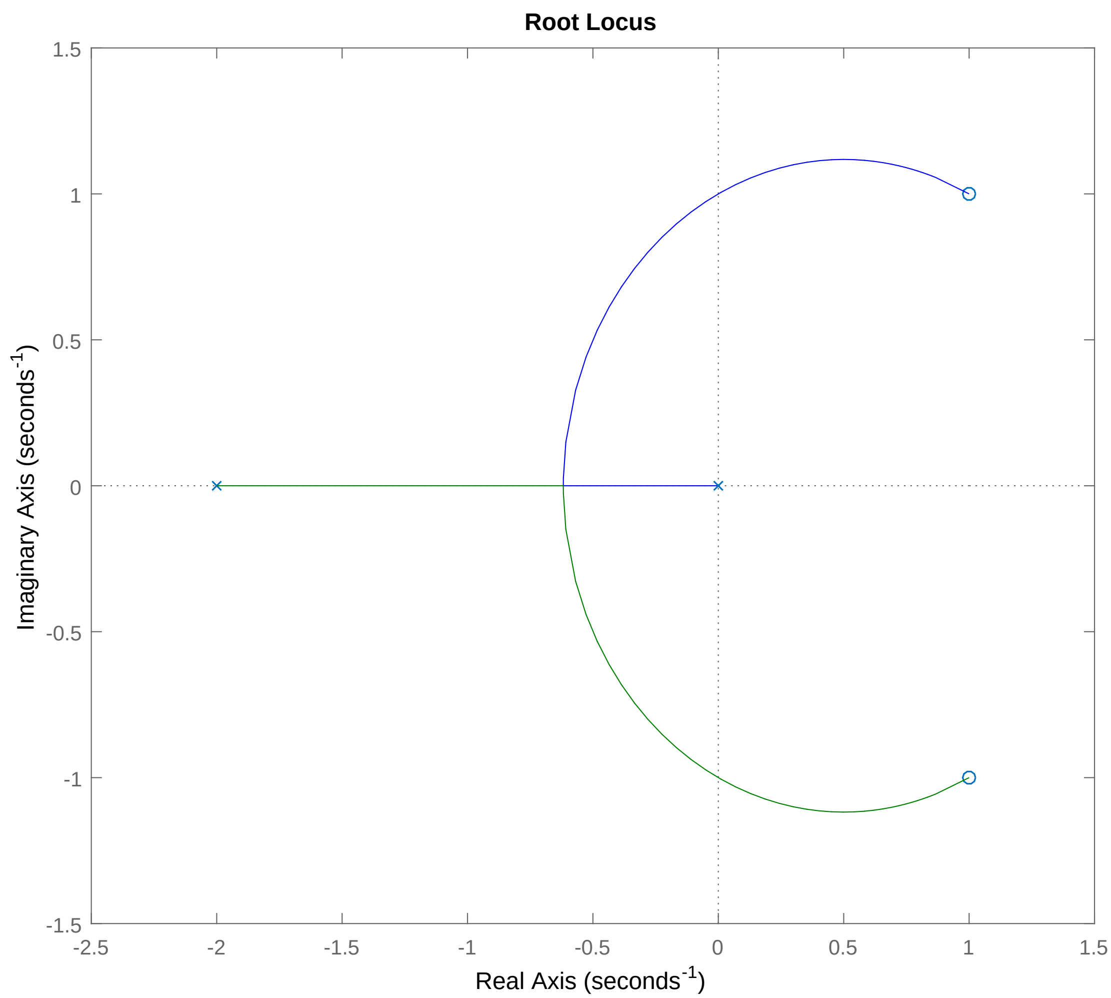
$$\frac{dK}{ds} = 0 \Rightarrow (2s+2)(s^2 - 2s + 2) = 0$$

$$\Rightarrow s_{1,2,3} = -1, -0.62, 1.62 \Rightarrow s_2 = -0.62 \text{ is the breakaway point}$$

4. Angles of arrival at the complex zeros:

$$\phi_{z_1} = 180 + 45 + \arctan(1/3) - 90 = 153.43\,\text{deg} \Rightarrow \phi_{z_2} = -153.43\,\text{deg}$$

5. $j\omega$ axis crossing: $K \approx 0.95, \omega \approx \pm 1$
6. Plot:


Root Locus
1.5
1
0.5
0
-0.5
-1
-1.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})

	Unit step input u(t)=1	Unit ramp input u(t)=t	Acceleration input u(t)=$t^2/2$
Type 0 systems	$\frac{1}{1+K_p}$ $K_p = G(0)$	∞	∞
Type 1 systems	0	$\frac{1}{K_v}$ $K_v = \lim_{s\to 0} sG(s)$	∞
Type 2 systems	0	0	$\frac{1}{K_a}$ $K_a = \lim_{s\to 0} s^2G(s)$

Rule 6 Asymptotes angles: RL branches ending at OL zeros at ∞ approach the asymptotic lines with angles:

$$\phi_q = \frac{(1+2q)180}{n_p - n_z} \text{deg}, \forall q = 0, 1, 2, \ldots, n_p - n_z - 1$$

Rule 7 Real-axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} Re(p_i) - \sum_{j=1}^{n_z} Re(z_j)}{n_p - n_z}$$

Rule 9 **Angle of Departure (AoD):** defined as the angle from a complex pole or Angle of Arrival (AoA) at a complex zero:

$$\text{AoD from a complex pole : } \phi_p = 180 - \sum_i \angle p_i + \sum_j \angle z_j,$$

$$\text{AoA at a complex zero : } \phi_z = 180 + \sum_i \angle p_i - \sum_j \angle z_j$$

- $\sum_i \angle p_i$ is the sum of all angles of vectors to a complex pole in question from other poles, $\sum_j \angle z_j$ is the sum of all angles of vectors to a complex pole in question from other zeros
- '$\angle$' denotes the angle of a complex number

Rule 10 Determine whether the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i$$

and finding the ω and K that solves the above equation. The value of ω you get is the frequency at which the RL crosses the imaginary y-axis and the K you get is the associated gain for the controller. You should obtain two equations (real = 0 and imaginary = 0) with two unknowns (K, ω). From there, you solve for K, ω pairs