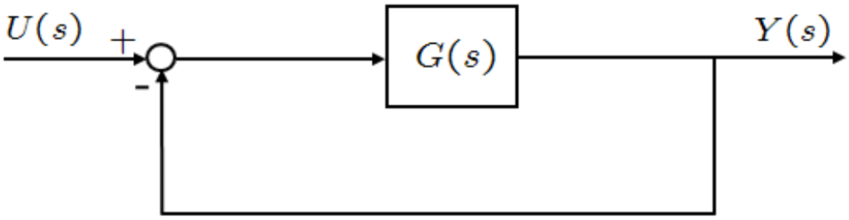
**EE 3413** **Exam II — Ahmad F. Taha** **Spring 2020**

Your Name:

Your Signature:

- **Exam duration:** 2 hours.
- This exam is open book, open notes, open laptops, open phones, open tablets, open pretty much everything.
- You can use MATLAB whenever you want to find roots for polynomials, to compute angles, to approximate things, to verify your calculations, but your handwritten solutions need to be extremely detailed. I will be grading based on the level of detail in your solution.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

| Question Number | Maximum Points | Your Score |
|---|---|---|
| 1 | 20 | |
| 2 | 30 | |
| 3 | 25 | |
| 4 | 25 | |
| ***Total*** | 100 | |
| ***Bonus*** | 15 | |



1. (20 total points) For the system shown in the above figure, assume that $G(s) = \frac{1}{s^2+4s+20}$. Answer the following questions.
   (a) (5 points) What is the closed-loop transfer function (CLTF)?

   **Solution:** CLTF $= \frac{G(s)}{1+G(s)} = \frac{1}{s^2+4s+21}$

   (b) (5 points) What are the poles of the CLTF?

   **Solution:** Piles are: $-2 \pm j4.123$

   (c) (5 points) Find the settling time of the CLTF via the 2% criterion, the maximum overshoot, and the peak time.

   **Solution:** $\omega_n = \sqrt{21}, \zeta = 0.43 \;\Rightarrow\; t_s = 2\,\text{sec},\;\; M_p = 21.78\%,\; T_p = 0.7619\,\text{sec}$

   (d) (5 points) Compute the steady-state error (SSE) **corresponding to a unit ramp input**. Use whatever tables you want.

   **Solution:**

$$E(s) = \frac{1}{1+G(s)}U(s) = \frac{1}{1+G(s)}\frac{1}{s^2} = \frac{s^s+4s+20}{s^2(s^2+4s+21)}.$$

   Hence,

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{4}{21} = 0.1904.$$

2. (30 total points) The characteristic polynomial (CP) of a system is given as follows:

$$1+KG(s)=1+K\frac{1}{s^3+5s^2+24s+20}=1+K\frac{1}{(s+1)(s^2+4s+20)}.$$

We want to sketch the root locus in this problem of the given CP.

(a) (5 points) Find the poles, zeros of the OLTF. Find the part of the real axis on the root locus. How many branches does the root locus have?

**Solutions:** Poles: $-1, -2 \pm 4j$. Root locus exists to the left of $-1$ on the axis. The root locus has three branches since $n_p = 3$.

(b) (5 points) Find the asymptotes of the root locus, including their angles ($\phi_q$) and point of intersection ($\sigma_a$). The formulae are given on the last page of the exam.

**Solutions:** angles are $\phi_q = (1+2q)60$, $q = 0,1,2 \quad \Rightarrow \phi_q = 60, 180, 300$.

Point of intersection: $\sigma_a = -5/3$

(c) (10 points) Find the break-in/breakaway points of the root locus.

**Solutions:** $P(s) = s^3+5s^2+24s+20+K = 0 \rightarrow K(s) = -(s^3+5s^2+24s+20)$.

Hence,

$$\frac{dK(s)}{ds} = 0 \Rightarrow -3s^2-10s-24 = 0 \Rightarrow s_{1,2} = -1.667 \pm 2.285j.$$

None of the roots are on the real axis, hence there is no break-in/break-way points.

(d) (5 points) Find the angle of departure from the complex poles (you can only find one of these angles, since the RL is symmetric).

**Solutions:** $\phi_{p_1=-2+4j} = 180-90-75.96 = 14.04\,\text{deg}, \;\rightarrow\; \phi_{p_2=-2-4j} = -14.04\,\text{deg}$.

(e) (5 points) Find the crossings with the $j\omega$ axis, and sketch the root locus.

**Solutions:** The crossings with the $j\omega$ axis can be found as follows:

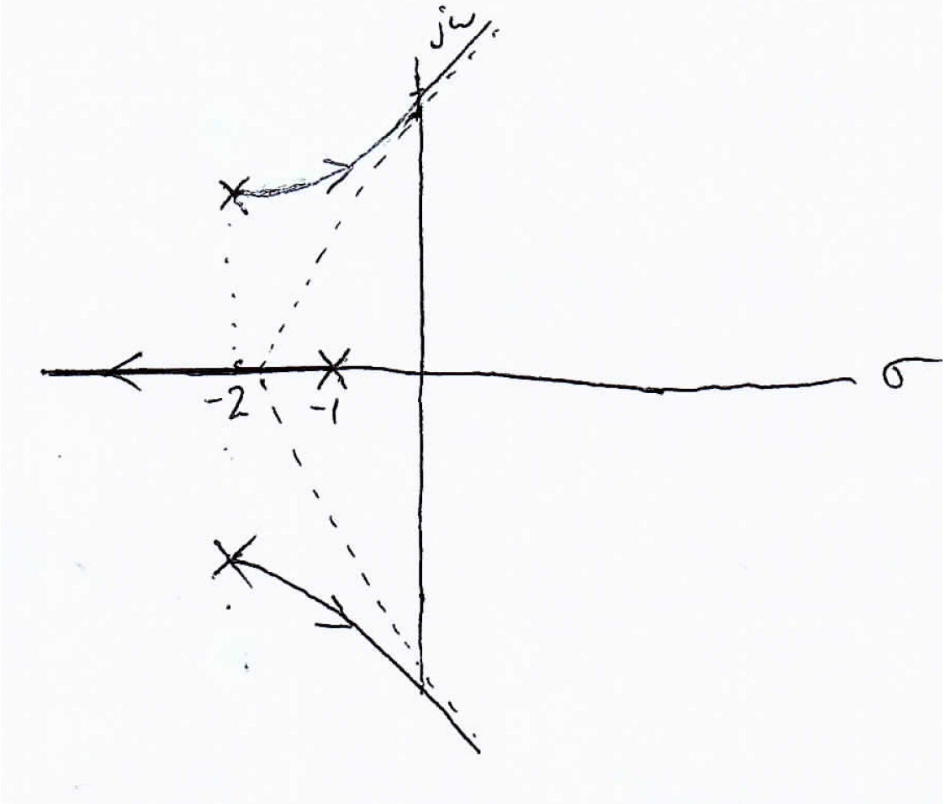
$$(j\omega)^3+5(j\omega)^2+24j\omega+20+K = 0+0j.$$

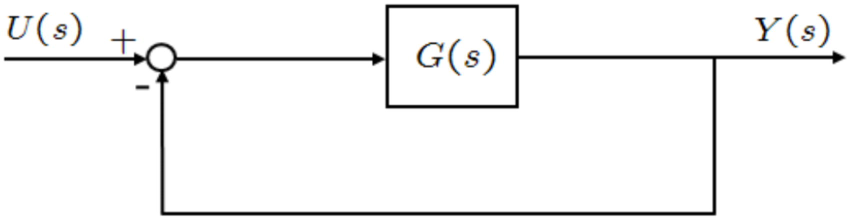
Hence,

$$20-5\omega^2+K+j(24\omega-\omega^3) = 0+0j.$$

The solution to this equation is

$$\omega^* = \pm\sqrt{24}, \quad K^* = 100.$$


jw
-2
-1
σ



3. (25 total points) For the above system, consider that $G(s) = \frac{K}{s(s+10)^2}$, $K \geq 0$. For this problem, you can use the SSE table immediately.
   (a) (5 points) Via the Routh array, find the range of $K$ such that the system is stable.
   **Solutions:** Transfer function is given as

$$CLTF = \frac{G(s)}{1+G(s)} = \frac{K}{s^3 + 20s^2 + 100s + K}.$$

   Applying the RHSC, we get $0 < K < 2000$.

   (b) (5 points) Given that $U_1(s) = \frac{1}{s}$, find the steady-state error $e_{ss1}$ corresponding to $U_1(s)$ in terms of $K$.
   **Solutions:** The system is Type 1, hence $e_{ss1} = 0$ for this unit step input.

   (c) (5 points) Given that $U_2(s) = \frac{1}{s^2}$, find the steady-state error $e_{ss2}$ corresponding to $U_2(s)$ in terms of $K$.
   **Solutions:** $K_v = \lim_{s \to 0} sG(s) = \frac{K}{100}$, $\rightarrow$ $e_{ss2} = \frac{1}{K_v} = \frac{100}{K}$.

   (d) (5 points) Given your solutions above, what is the overall SSE corresponding ($e_{ss}$) to an input $U(s) = 4U_1(s) + 5U_2(s)$? Your answer should also be in terms of $K$ and you should use your solutions above (a,b).
   **Solutions:** Since the system is linear, then we can write

$$e_{ss} = 4(e_{ss1}) + 5(e_{ss2}) = \frac{500}{K}.$$

   (e) (5 points) For (d) above, we want the overall SSE to be equal to or less than 0.3 for the above given $U(s)$, i.e., $e_{ss} \leq 0.3$. Find the minimum value of $K$ (i.e., $K_{min}$) that satisfies this requirement. Is this value possible?
   **Solutions:** $e_{ss} < 0.3 \Rightarrow 500/K < 0.3 \Rightarrow K > 500/0.3 = 1666.67 = K_{\min}$.

4. (25 total points) The characteristic polynomial of a closed-loop system is given by:

$$(2+K)s^2 + (2-4K)s + 5K = 0.$$

(a) (25 points) Plot the root locus. You should follow **all the steps we discussed in class**.

**Solutions:**

1. First, the characteristic polynomial can be written as

$$(2+K)s^2 + (2-4K)s + 5K = 0 \;\equiv\; 1 + K\frac{s^2-4s+5}{2s^2+2s} = 0.$$

2. Second, the poles and zeros are: $p_{1,2} = \{0, -1\}, z_{1,2} = 2 \pm j$.
3. Third, the RL on the axis is only between 0 and -1.
4. Fourth, the RL has $n_p = 2$ branches.
5. Fifth, the breakaway points are determined via first computing $K(s)$ as $K(s) = \dfrac{-2s^2-2s}{s^2-4s+5}$. Then setting $K'(s) = 0$, you will get $s_{1,2} = \{-0.4142, 2.414\}$. The point $s_2$ is not part of the root locus, whereas $s_1$ is a part of the RL.
6. Sixth, the point $s_1$ is a break-away point seeing that $K''(s_1) = -0.6066 < 0$.
7. Seventh, the $j\omega$ axis cross is determined via solving for $K$ and $\omega$ from

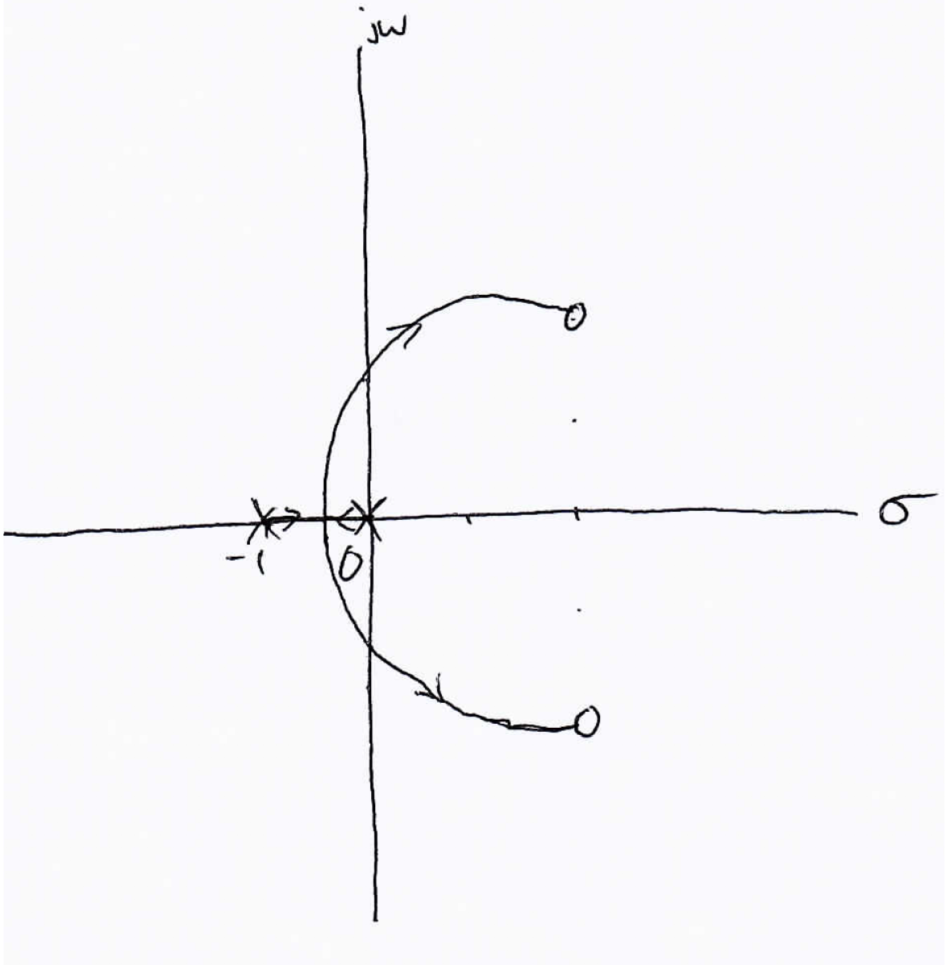
$$(2+K)(j\omega)^2 + (2-4K)(j\omega) + 5K = 0 + 0j.$$

This results in $K^* = 0.5$ which corresponds with $\omega^* = \pm\sqrt{\frac{5}{3}} = \pm 1.29$.

8. Eighth, angle of arrival at the complex zeros are

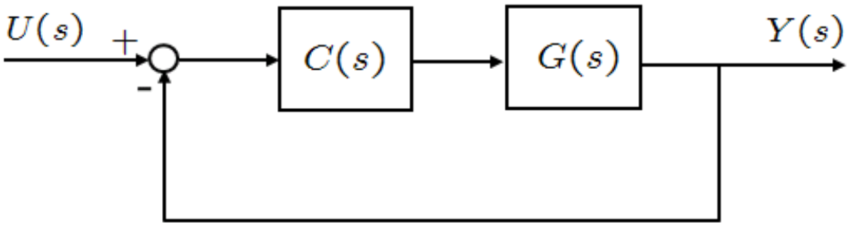
$$\phi_{z_1=2+j} = 180 + \sum_i \angle p_i - \sum_i \angle z_i = 180 + 26.565 + 18.43 - 90 = 135^\circ,\;\; \phi_{z_2=2-j} = -135^\circ.$$

9. Plot is given on the next page.


jw
σ
-1
0

5. (15 total points) **[Bonus Question]** For the system shown in the below figure, assume that:

$$G(s) = \frac{1}{s^4 + 10s^3 + 20s^2 + 20s - 3}\,, \; C(s) = 13 + \frac{K}{s}\,, \; K \geq 0.$$



   (a) (15 points) Find the CLTF, then using the Routh array criterion, find the range of $K$ such that the CLTF is stable.

**Solutions:** The CLTF is given as

$$\text{CLTF} = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{13s + K}{s^5 + 10s^4 + 20s^3 + 20s^2 + 10s + K}.$$

We now formulate the Routh array as

$$\begin{array}{c||c|c|c}
s^5 & 1 & 20 & 10 \\
s^4 & 10 & 20 & K \\
s^3 & 18 & 10 - 0.1K & 0 \\
s^2 & 14.44 + \dfrac{K}{18} & K & 0 \\
s & \Omega(K) & 0 & 0 \\
0 & K & 0 & 0
\end{array}$$

where

$$\Omega(K) = 10 - 0.1K - \frac{18K}{14.44 + \dfrac{K}{18}}.$$

Here, we first require that

$$14.44 + \frac{K}{18} > 0, \quad \Rightarrow \quad K > -260$$

which is already satisfied seeing that $K \geq 0$ by default. In addition, we require that

$$\Omega(K) = 10 - 0.1K - \frac{18K}{14.44 + \dfrac{K}{18}} > 0$$

which can be written equivalently as

$$P(K) = 0.0555K^2 + 18.88K - 144.4 < 0.$$

For this polynomial $P(K)$ to be negative, we need

$$(K + 347.66)(K - 7.48) < 0$$

which happens when

$$-347.66 < K < 7.48.$$

Since $K > 0$, we conclude that for the CLTF to be stable, the range for the range $K$ is

$$0 < K < 7.48.$$