

Module 02

Laplace Transforms, Transfer Functions & ODEs

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Module 02 Outline

- 1 We will introduce Laplace Transforms
- 2 Discuss their importance
- 3 Properties and definitions
- 4 Use them to solve ODEs
- 5 Understand partial fraction expansion
- 6 Transfer Functions... *They're Imaginary...*
- 7 Examples, Examples!
 - Reading material: Dorf & Bishop, Sections 2.4 & 2.5

Laplace Transform: Basic Definition

- **Laplace Transform:** takes a function of t (time) to a function of a complex variable s (frequency)
- Given a function in time ($t \geq 0$), $f(t)$, we want to apply this transformation:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- Above integral might be undefined for some s
- **Abscissa of convergence:** smallest σ such that for all s with $\text{Re}(s) \geq \sigma$, the integral above converges

Example 1: Step, Ramp, and Exponential Signals

What is the LT of $f(t) = 5 \forall t \geq 0$?

What is the LT of $f(t) = 2t \forall t \geq 0$?

What is the LT of $f(t) = e^{-at} \forall t \geq 0$?

Laplace Transform Table

- Integration can sometimes be tedious
- And we are often too lazy to do it
- Always look at the Laplace transform table and compare to what you have
- Sometimes, you have to tweak your function to fit with the given transforms

Laplace Transform Table — 1

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. \sqrt{t}	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$

Laplace Transform Table — 2

19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2 - b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2 - b^2}$
23. $t^n e^{at}$, $n=1, 2, 3, \dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-cs}
27. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	28. $u_c(t) g(t)$	$e^{-cs} \Omega\{g(t+c)\}$
29. $e^{at} f(t)$	$F(s-c)$	30. $t^n f(t)$, $n=1, 2, 3, \dots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t} f(t)$	$\int_s^\infty F(u) du$	32. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$	34. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
37. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$		

Laplace Transform Properties — Linearity

- Laplace transform, by definition, is a linear mapping (transformation)
- In other words:

$$\mathcal{L}[\alpha_1 f_1(t) + \alpha_2 f_2(t)] = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

- Can you prove it? It's so easy
- **Proof:**
- Example: $\mathcal{L}[5 \cdot 1^+(t) + 2e^{-2}(t)] = ?$

Laplace Transform Properties — Differentiation

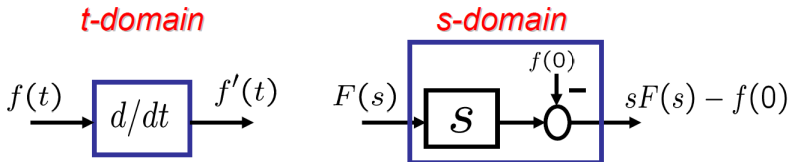
- What is the Laplace transform of a derivative of a function, $f'(t)$?

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

- $f(0)$ is the initial conditions of the function $f(t)$ at $t = 0$
- Can you prove it? It's easy — you need to know integration by parts
- Example: if $f(t) = \cos(2t)$, what is $\mathcal{L}[f'(t)]$?
- Higher order differentiation:

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- Illustration



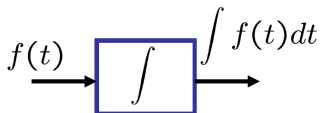
Laplace Transform Properties — Integration

- What is the Laplace transform of an integral of a function?

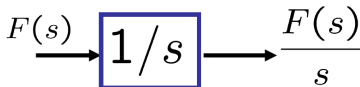
$$\mathcal{L} \left[\int_0^t f(\tau) d\tau \right] = \frac{F(s)}{s}$$

- Can you prove it? Integration by parts, again!
- **Proof:**
- Illustration

t-domain



s-domain



Laplace Transform Properties — Final Value Theorem

- Consider $F(s) = \frac{N(s)}{D(s)}$
- Poles of $F(s)$: roots($D(s)$)
- Zeros of $F(s)$: roots($N(s)$)
- * **Final Value Theorem:**

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \text{ if all poles of } sF(s) \text{ are in LHP}$$

- Example 1:

$$F(s) = \frac{5}{s(s^2 + s + 2)} \Rightarrow \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \frac{5}{s^2 + s + 2} = \frac{5}{2}$$

- Example 2:

$$F(s) = \frac{4}{s^2 + 4} \Rightarrow \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \frac{4s}{s^2 + 4} = 0 \text{ WRONG!!!}$$

FVL Example



- With all zero initial conditions for $y(t)$ and $u(t)$, system is governed by this second order ODE:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\dot{u}(t) + u(t)$$

- Using FVT, find $\lim_{t \rightarrow \infty} y(t)$ if $u(t) = 1$
- **Solution:**

Laplace Transform Properties — Initial Value Theorem

- Consider $F(s) = \frac{N(s)}{D(s)}$

- * **Initial Value Theorem:**

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s) \text{ if the limit exists}$$

- Note: In this theorem, it does not matter if pole location is in LHP or not
- Example 1:

$$F(s) = \frac{5}{s(s^2 + s + 2)} \Rightarrow \lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{5}{s^2 + s + 2} = 0$$

- Example 2:

$$F(s) = \frac{4}{s^2 + 4} \Rightarrow \lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{4s}{s^2 + 4} = 0$$

Laplace Transform Properties — Convolution

- We all hate convolutions, right?
- What is convolution anyway?

$$f_1(t) * f_2(t) = \int_0^t f_1(t - \tau) \cdot f_2(\tau) d\tau = \int_0^t f_2(t - \tau) \cdot f_1(\tau) d\tau$$

- What is the Laplace transform of $f_1 * f_2$?

$$\mathcal{L}[f_1(t) * f_2(t)] = \mathcal{L}[f_1(t)] \cdot \mathcal{L}[f_2(t)] = F_1(s) \cdot F_2(s)$$

- Note: $F_1(s)F_2(s) \neq \mathcal{L}[f_1(t)f_2(t)]$
- Laplace transform of convoluted functions is a smart way to run away from doing convolutions

Inverse Laplace Transform

- Given a function in time ($t \geq 0$), $f(t)$, Laplace Transform is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- What if I have $F(s)$? Can I obtain $f(t)$ back? Yes.You.Can!
- Inverse Laplace Transform:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} dt$$

- The above integral is often very hard to solve, so it's easier to look at the table and figure things out
- Alternative: partial fraction expansion

What are Transfer Functions?



- * **TFs:** *a mathematical representation to describe relationship between inputs and outputs of the physics of a system, i.e., of the differential equations that govern the motion of bodies, for example*
- **Input:** always defined as $u(t)$ —called control action
- **Output:** always defined as $y(t)$ —called measurement or sensor data
- TF relates the derivatives of $u(t)$ and $y(t)$
- Why is that important? Well, think of $\sum F = ma$
- F above is the input (exerted forces), and the output is the acceleration, a
- Give me the equations, please...

Construction of Transfer Functions



- For linear systems, we can often represent the system dynamics through an n th order ordinary differential equation (ODE):

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_0y(t) = u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + b_{m-2}u^{(m-2)}(t) + \cdots + b_0u(t)$$

- The $y^{(k)}$ notation means we're taking the k th derivative of $y(t)$
- Typically, $m > n$
- Given that ODE description, we can take the LT (assuming zero initial conditions for all signals):

$$F(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

What are Transfer Functions?



- Given this TF:

$$F(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

- For a given control signal $u(t)$ or $U(s)$, we can find the output of the system, $y(t)$, or $Y(s)$
- But to do that, we need to take the inverse Laplace transform
- We can do that using partial fraction expansion
- Remember:** *TFs are imaginary, $Y(s)$ often means nothing*

Partial Fraction Expansion

- **Objective 1:** find the inverse Laplace transform of $F(s)$ ($f(t) = \mathcal{L}^{-1}[F(s)]$) given that

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_n}, \quad n \geq m$$

- For linear systems, $N(s)$ and $D(s)$ **are always polynomials**
- Typically, $\text{order}(D(s)) > \text{order}(N(s))$, i.e., $n \geq m$
- **Objective 1.1:** write $F(s) = \frac{N(s)}{D(s)}$ in terms of known expressions from the LTs table
- Three major cases—roots of $D(s)$ are: (A) distinct, (B) equal, (C) complex

Case A — Distinct $D(s)$ Roots

- Distinct roots for $D(s)$ means that we can write $F(s)$ as:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{a_0(s - p_1)(s - p_2)\dots(s - p_n)}$$

- Since $n \geq m$, the above form can simply be written as:

$$F(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \dots + \frac{c_n}{s - p_n}$$

- How will this form help me find $f(t)$?
- Well, that's easy: $f(t) = \mathcal{L}^{-1}[F(s)] = c_1e^{p_1t} + \dots + c_n e^{p_nt}$
- Assume that p_i 's are given
- Task: find the so-called *residues* c_i 's for poles p_i 's
- **Solution:** $c_i = (s - p_i)F(s) \Big|_{s=p_i}$

Case A — Example

- Find $f(t)$ if $F(s) = \frac{1}{s^2 + 5s + 6}$
- Roots of $s^2 + 5s + 6$ are: $p_{1,2} = -3, -2$
- Hence, $F(s) = \frac{1}{(s+3)(s+2)} = \frac{c_1}{s+3} + \frac{c_2}{s+2}$
- Using the method in the previous slide:

$$c_1 = (s+3)F(s) \Big|_{s=-3} = -1$$

$$c_2 = (s+2)F(s) \Big|_{s=-2} = 1$$

- Thus, $F(s) = \frac{1}{(s+3)(s+2)} = \frac{-1}{s+3} + \frac{1}{s+2}$
- Can you find $f(t)$ now? Of course you can

Case B — Repeated Roots

- For this case, assume that $D(s)$ includes $(s - p_1)^k$, where k is the multiplicity of pole p_1

- Find $f(t)$ if $F(s) = \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

- To find A, C , use the Case A method:

$$A = (s+1)F(s) \Big|_{s=-1} = 1, \quad C = (s+2)^2 F(s) \Big|_{s=-2} = -1$$

- To find B , substitute for $s = \text{RandomNumber}$ where $\text{RandomNumber} \neq -1, -2$
- Set $s = 0$, then $B = -1$
- Given A, B, C find $f(t)$ using the table

Case C — Imaginary Roots

- What if roots are imaginary? Remember that complex roots come in complex conjugates

- Consider that $F(s) = \frac{1}{s(s^2 + 2s + 2)}$

- Roots of $s^2 + 2s + 2$ are complex conjugates since $\Delta = 4 - 8 = -4 < 0$

- We can write

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

- Find A using the Case A method: $A = 0.5$
- How can we find B, C ? Many options, but basically, you have to solve a system of two equations, two unknowns
- **Solution:** $B = -0.5, C = -1$

Case C — Imaginary Roots (Cont'd)

- We now have:

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{0.5}{s} + \frac{-0.5s - 1}{s^2 + 2s + 2}$$

- What is $f(t)$?
- From the LT table:

$$\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2 + b^2}\right] = e^{at} \sin(bt), \quad \mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2 + b^2}\right] = e^{at} \cos(bt)$$

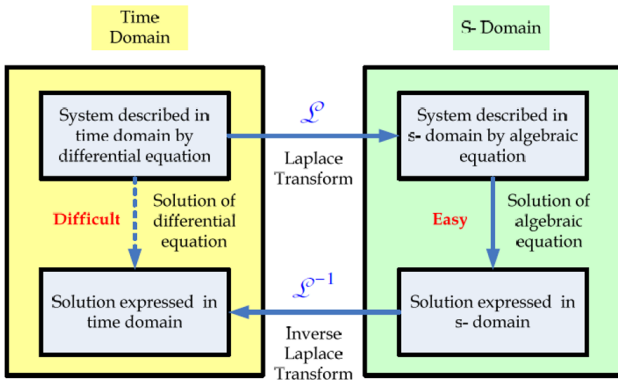
- Second part of $F(s)$ can be written as:

$$\begin{aligned} \frac{-0.5s - 1}{s^2 + 2s + 2} &= -0.5 \frac{s + 2}{s^2 + 2s + 2} = -0.5 \frac{(s + 1) + 1}{(s + 1)^2 + 1^2} \\ &= -0.5 \frac{(s + 1)}{(s + 1)^2 + 1^2} - 0.5 \frac{1}{(s + 1)^2 + 1^2} \end{aligned}$$

- We can now go back to the ILT table and find $f(t)$
- $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] = e^{-t} \sin(t)$, $\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1^2}\right] = e^{-t} \cos(t)$

Why? Why Not?

- Solving ODEs using time-domain techniques can be very challenging
- Laplace transforms and PFEs offer an easy way to solve ODEs
- Basic idea is as follows:



Example 1

- Solve this ODE, i.e., find $y(t)$:

$$\ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1$$

- Remember this property:

$$\mathcal{L} \left[f^{(n)}(t) \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

- **Solution:**

- 1 Take Laplace Transform
- 2 Find $Y(s)$
- 3 Apply PFE for $Y(s)$ —find the residues
- 4 Use the ILT to find $y(t)$
- 5 Answer: $y(t) = 1.5e^t - 0.5e^{-t} - t$

Example 2

- Solve this ODE, i.e., find $y(t)$:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 5, \quad y(0) = -1, \dot{y}(0) = 2$$

- **Solution:**

- 1 Take Laplace Transform
- 2 Find $Y(s)$
- 3 Apply PFE for $Y(s)$ —find the residues
- 4 Use the ILT to find $y(t)$
- 5 Answer: $y(t) = 1.5e^{-2t} - 5e^{-t} + 2.5$

MATLAB Demo

- Most of what we learned today can be easily tested on MATLAB
- Let's start with the basics
- First, you need to understand the language of MATLAB
- Symbolic toolbox: provides functions for solving and manipulating symbolic math expressions
- Example: `syms x y`—for LTs, define `syms s t`
- To find the Laplace transform, use the command `laplace`
- To find the inverse Laplace transform, use the command `ilaplace`

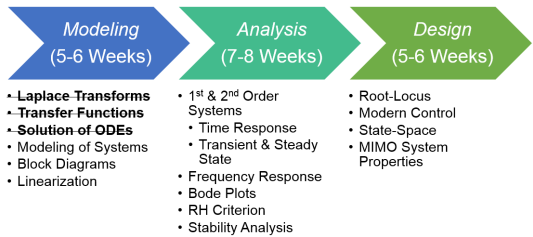
MATLAB Examples

- `syms s t`
- `ans1=laplace(t*exp(3*t))`
- `iplace(ans1)`
- `ans2=laplace(t*t*cos(5*t)*exp(3*t))`
- `iplace(ans2)`
- Let's try the PFE, command: `residue`
- `[R,P,K] = residue(B,A)`

Where are we now?

In this module, we learned:

- How to analytically compute Laplace transforms
- How to be lazy and look at the table—unless you are a genius
- Final and initial value theorems, because computing analytical limits is too mainstream
- Inverse Laplace transforms and PFEs
- Solving ODEs using LTs
- Where are we now?



Questions And Suggestions?



Thank You!

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IFF you want to know more 😊