

Module 03

Modeling of Dynamical Systems

Ahmad F. Taha

EE 3413: Analysis and Design of Control Systems

Email: ahmad.taha@utsa.edu

Webpage: <http://engineering.utsa.edu/ataha>



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Module 3 Outline

- 1 Physical laws and equations
- 2 Transfer function model
- 3 Model of electrical systems
- 4 Model of mechanical systems
- 5 Examples
 - Reading material: Dorf & Bishop, Section 2.3

Physical Laws and Models

- By definition, dynamical systems **are dynamic** because they change with time
- Change in the sense that their intrinsic properties evolve, vary
- Examples: coordinates of a drone, speed of a car, body temperature, concentrations of chemicals in a centrifuge
- Physicists and engineers like to represent dynamic systems with equations
- Why? Well, the answer is fairly straightforward
- Dynamic model often means a differential equations

Physical Laws

- For many systems, it's easy to understand the physics, and hence the math behind the physics
 - Examples: circuits, motion of a cart, pendulum, suspension system
- For the majority of dynamical systems, the actual physics is complex
- Hence, it can be hard to depict the dynamics with ODEs
 - Examples: human body temperature, thermodynamics, spacecrafts
- This illustrates the needs for *models*
- **Dynamic system model:** a mathematical description of the actual physics

What are Transfer Functions?



- * **TFs:** *a mathematical representation to describe relationship between inputs and outputs of the physics of a system, i.e., of the differential equations that govern the motion of bodies, for example*
- **Input:** always defined as $u(t)$ —called control action
- **Output:** always defined as $y(t)$ —called measurement or sensor data
- TF relates the derivatives of $u(t)$ and $y(t)$
- Why is that important? Well, think of $\sum F = ma$
- F above is the input (exerted forces), and the output is the acceleration, a
- Give me the equations, please...

Construction of Transfer Functions



- For linear systems, we can often represent the system dynamics through an n th order ordinary differential equation (ODE):

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_0y(t) = u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + b_{m-2}u^{(m-2)}(t) + \cdots + b_0u(t)$$

- The $y^{(k)}$ notation means we're taking the k th derivative of $y(t)$
- Typically, $m > n$
- Given that ODE description, we can take the LT (assuming zero initial conditions for all signals):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

What are Transfer Functions?

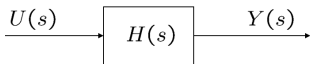


- Given this TF:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

- For a given control signal $u(t)$ or $U(s)$, we can find the output of the system, $y(t)$, or $Y(s)$
- Impulse response:** defined as $h(t)$ —the output $y(t)$ if the input $u(t) = \delta(t)$
- Step response:** the output $y(t)$ if the input $u(t) = 1^+(t)$
- For any input $u(t)$, the output is: $y(t) = h(t) * u(t)$
- But...Convolutions are nasty...Who likes them?

TFs of Generic LTI Systems

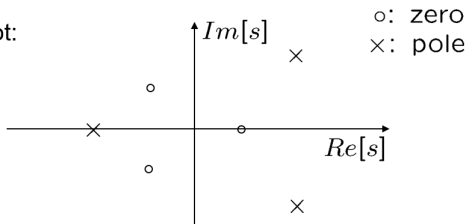


- So, we can take the Laplace transform: $Y(s) = H(s)U(s)$
- Typically, we can write the TF as:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

- Roots of numerator are called the **zeros** of $H(s)$ or the system
- Roots of the denominator are called the **poles** of $H(s)$

Pole zero plot:



Example

Given: $H(s) = \frac{2s + 1}{s^3 - 4s^2 + 6s - 4}$

- **Zeros:** $z_1 = -0.5$
- **Poles:** solve $s^3 - 4s^2 + 6s - 4 = 0$, use MATLAB's roots command
- * $\text{poles} = \text{roots}[1 \ -4 \ 6 \ -4]$; $\text{poles} = \{2, 1 + j, 1 - j\}$
- **Factored form:**

$$H(s) = 2 \frac{s + 0.5}{(s - 2)(s - 1 - j)(s - 1 + j)}$$

Analyzing Generic Physical Systems

Seven-step algorithm:

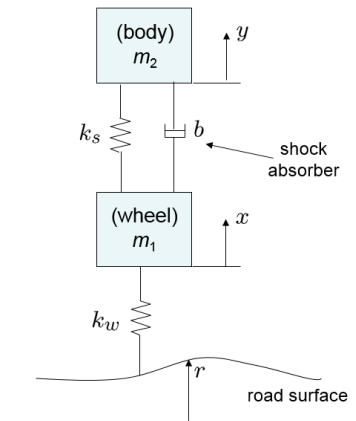
- 1 Identify dynamic variables, inputs (u), and system outputs (y)
- 2 Focus on one component, analyze the dynamics (physics) of this component
 - How? Use Newton's Equations, KVL, or thermodynamics laws...
- 3 After that, obtain an n th order **ODE**:

$$\sum_{i=1}^n \alpha_i y^{(i)}(t) = \sum_{j=1}^m \beta_j u^{(j)}(t)$$

- 4 Take the Laplace transform of that **ODE**
- 5 Combine the equations to eliminate internal variables
- 6 Write the transfer function from input to output
- 7 For a certain control $U(s)$, find $Y(s)$, then $y(t) = \mathcal{L}^{-1}[Y(s)]$

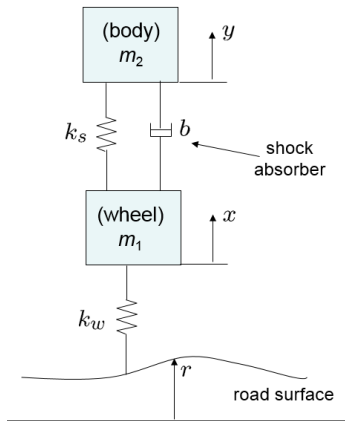
Active Suspension Model

- Each car has 4 active suspension systems (on each wheel)
- System is nonlinear, but we consider approximation. **Objective?**
- **Input:** road altitude $r(t)$ (or $u(t)$), **Output:** car body height $y(t)$



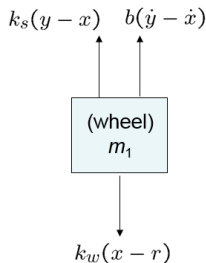
Active Suspension Model — Equations for 1 Wheel

- We only consider one of the four systems
- System has many components, most important ones are: body (m_2) & wheel (m_1) weights

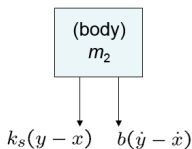
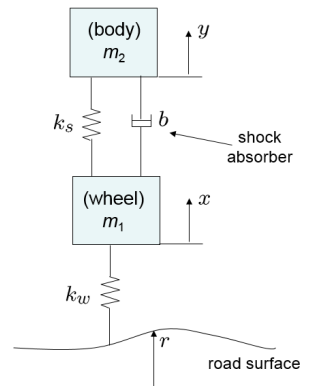


By Newton's Second Law

$$m_1 \ddot{x} = k_s(y - x) + b(\dot{y} - \dot{x}) - k_w(x - r)$$



Active Suspension Model — Equations for Car Body



By Newton's Second Law

$$m_2 \ddot{y} = -k_s(y - x) - b(\dot{y} - \dot{x})$$

- We now have 2 equations depicting the car body and wheel motion
- Objective: find the TF relating output ($y(t)$) to input ($r(t)$)
- What is $H(s) = \frac{Y(s)}{R(s)}$?

Active Suspension Model — Transfer Function

- **Differential equations (in time):**

$$m_1\ddot{x}(t) = k_s(y(t) - x(t)) + b(\dot{y}(t) - \dot{x}(t)) - k_w(x(t) - r(t))$$

$$m_2\ddot{y}(t) = -k_s(y(t) - x(t)) - b(\dot{y}(t) - \dot{x}(t))$$

- **Take Laplace transform given zero ICs:**

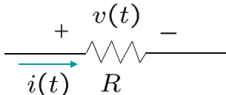
– **Solution:**

- Find $H(s) = \frac{Y(s)}{R(s)}$

– **Solution:**

Basic Circuits Components

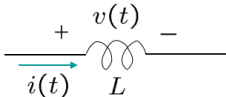
resistor



$$v(t) = Ri(t)$$

$$V(s) = RI(s) \Rightarrow \frac{V(s)}{I(s)} = R$$

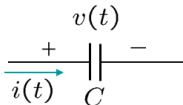
inductor



$$v(t) = L \frac{di(t)}{dt}$$

$$V(s) = LsI(s) \Rightarrow \frac{V(s)}{I(s)} = Ls$$

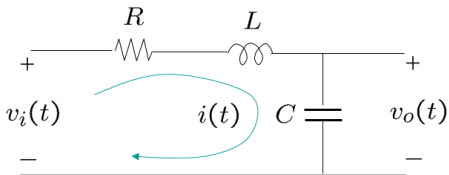
capacitor



$$i(t) = C \frac{dv(t)}{dt}$$

$$I(s) = CsV(s) \Rightarrow \frac{V(s)}{I(s)} = \frac{1}{Cs}$$

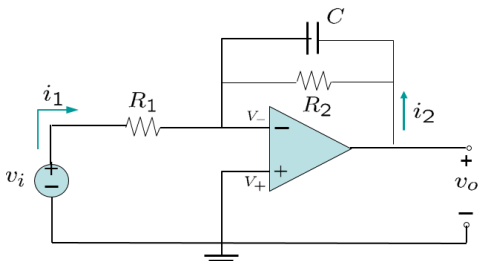
Basic Circuits — RLCs & Op-Amps



$v_i(t)$: input

$v_o(t)$: output

Transfer function $\frac{V_o(s)}{V_i(s)}$

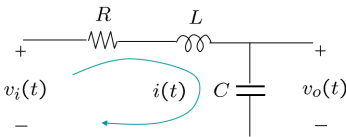


$v_i(t)$: input

$v_o(t)$: output

Transfer function $\frac{V_o(s)}{V_i(s)}$

TF of an RLC Circuit — Example



Objective: Find TF

$v_i(t)$: input

$v_o(t)$: output

Transfer function $\frac{V_o(s)}{V_i(s)}$

- Apply KVL (assume zero ICs):

$$v_i(t) = Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(\tau) dt$$

$$v_o(t) = \frac{1}{C} \int i(\tau) dt$$

- Take LT for the above differential equations:

$$V_i(s) = RI(s) + LsI(s) + \frac{1}{Cs} I(s)$$

$$V_o(s) = \frac{1}{Cs} I(s) \Rightarrow I(s) = CsV_o(s)$$

$$\Rightarrow \boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Generic Circuit Analysis

s-Domain Circuit Analysis

**Time domain
(t domain)**



Linear
Circuit

Differential
equation

Classical
techniques

Response
waveform

Laplace Transform

\mathcal{L}

Laplace Transform

\mathcal{L}

Inverse Transform

\mathcal{L}^{-1}

**Complex frequency
domain (s domain)**



Transformed
Circuit

Algebraic
equation

Algebraic
techniques

Response
transform

Dynamic Models in Nature

- Predator-prey equations are 1st order non-linear, ODEs
- Describe the dynamics of biological systems where 2 species interact
- One species as a predator and the other as a prey
- Populations change through time according to these equations:

$$\dot{x}(t) = \alpha x(t) - \beta x(t)z(t)$$

$$\dot{z}(t) = \delta x(t)z(t) - \gamma z(t)$$

- $x(t)$: # of preys (e.g., rabbits)
- $z(t)$: # of predators (e.g., foxes)
- $\dot{x}(t), \dot{z}(t)$: growth rates of the 2 species—function of time, t
- $\alpha, \beta, \gamma, \delta$: +ve real parameters depicting the interaction of the species

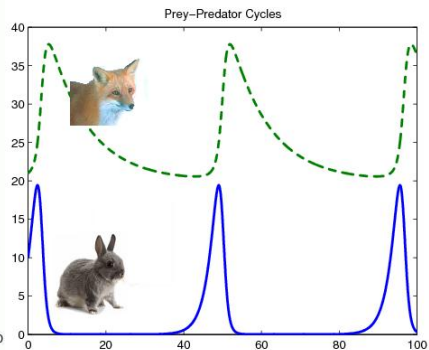
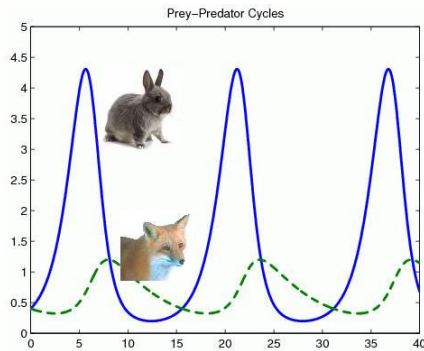
Mathematical Model

$$\dot{x}(t) = \alpha x(t) - \beta x(t)z(t)$$

$$\dot{z}(t) = \delta x(t)z(t) - \gamma z(t)$$

- Prey's population grows exponentially ($\alpha x(t)$)—why?
- Rate of *predation* is assumed to be proportional to the rate at which the predators and the prey meet ($\beta x(t)z(t)$)
- If either $x(t)$ or $z(t)$ is zero then there can be no predation
- $\delta x(t)z(t)$ represents the growth of the predator population
- No prey \Rightarrow no food for the predator $\Rightarrow z(t)$ decays
- Is there an equilibrium? What is it?
- No transfer function here...Why?
- Check [Wiki](#)

Example: impact of different parameters α, β, \dots



Dynamics in Epidemiology

- **Epidemiology:** The branch of medicine that deals with the incidence, distribution, and possible control of diseases and other factors relating to health
- In the past 10 years, mathematicians, biologists, and physicists studied mathematical models of epidemics
- Why is that important?
- Various models focus on different things:
 - SIR Model: **S** for the number susceptible, **I** for the number of infectious, and **R** for the number recovered
 - SIS Model: Infections like cold and influenza, do not possess lasting immunity
 - SEIR: **E** for exposed
 - MSIR: **M** stands for maternally-derived immunity
 - SEIS and many, many more

SIR Model

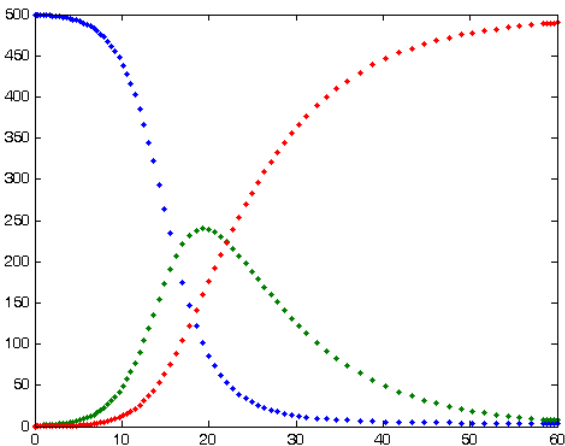


- Here, we present the dynamic model for the SIR model
- We take flu as an example of the SIR model
- Define variable $S(t)$, $I(t)$, $R(t)$ representing the number of people in each category at time t . The SIR model can be written as

$$\begin{aligned}\frac{dS}{dt} &= -\frac{\beta IS}{N} \\ \frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I \\ \frac{dR}{dt} &= \gamma I.\end{aligned}$$

- N is the total number of people, with $S(t) + I(t) + R(t) = N$
- The force of infection F can be written as $F = \beta I/N$
- β is the contact rate, and γ is the transition rate (rate of recovery)

So who do these quantities vary?



Blue represents **Susceptible**, Green represents **Infected**, and Red represents the **Recovered** population.

Nonlinear Dynamical Systems

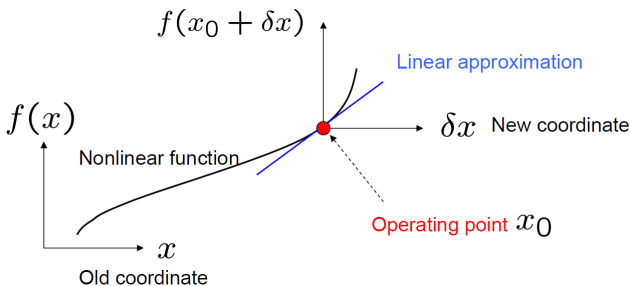
- Let's face it: most dynamical systems are **nonlinear**
- Nonlinearities can be seen in the ODEs, e.g.:

$$\dot{y}(t) + \dot{y}(t)\ddot{y}(t) + \cos(y(t)) = 2u(t) + \arctan(e^{\cos(u(t))})$$

- Examples: electromechanical systems, electronics, hydraulic systems, thermal, etc...
- *Why do we hate nonlinear systems?*
 - Well, because we cannot solve ODEs tractably if they are not linear
- I mean we can, but they're hard—and remember, we're lazy
- **Solution: linearize** nonlinear equations
- Btw...most nonlinear systems are linear for a *short period of time*
- So, it's legit to linearize for a *short period of time*

Linearization — The Main Idea

- Linearization is one of the most important techniques in control theory
- Without it, all our analysis of nonlinear systems becomes pointless
- First, let's assume that a nonlinear system is **linearized around an operating point**
- **Operating point** is often called **equilibrium point**
- Main idea:



Linearization — The Simple Math

- Nonlinear equation (or system): $\dot{x}(t) = f(x, u)$
- **Equilibrium points:** u_e, x_e
- Equilibrium **deviation** : $\delta u(t) = u(t) - u_e, \delta x(t) = x(t) - x_e$
- Taylor series expansion around u_e, x_e :

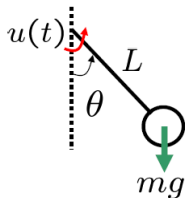
$$\dot{x}(t) \approx f(x_e, u_e) + (\delta x(t)) \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_e, u_e} + (\delta u(t)) \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_e, u_e}$$

- Hence:

$$\delta \dot{x}(t) \approx (\delta x(t)) \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_e, u_e} + (\delta u(t)) \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_e, u_e}$$

- This relationship is a linear one between δx and δu

Linearization — Example



- Pendulum motion:

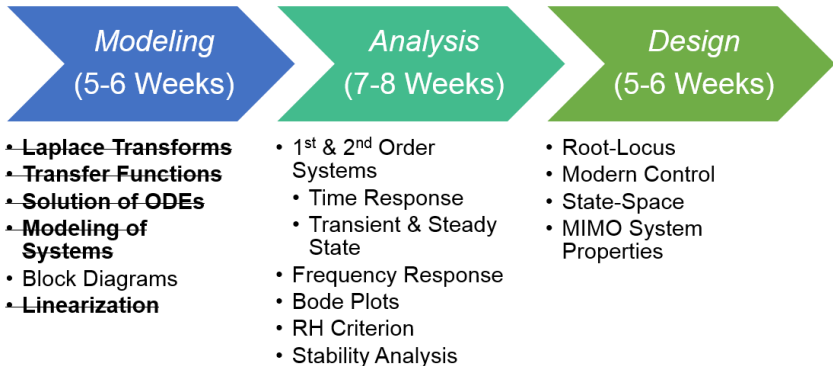
$$f(x, u) = -\frac{g}{L} \sin(x(t)) + \frac{1}{mL^2} u(t)$$

- $x(t)$: angle (θ), $u(t)$: force
- Given equilibrium points: $u_e = 0, x_e = \pi$
- Taylor series expansion around $0, \pi$:

$$\delta f(\delta x, \delta u) \approx \frac{g}{L} \delta x(t) + \frac{1}{mL^2} \delta u(t)$$

- This relationship is a linear one between δx and δu : only valid in the *vicinity of the equilibrium point*

Roadmap Revisited



Questions And Suggestions?



Thank You!

Please visit

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IFF you want to know more 😊