

Module 05

System Analysis & First and Second Order Dynamical Systems

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EE 3413: Analysis and Design of Control Systems

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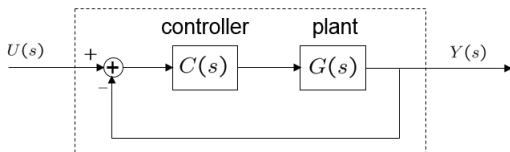
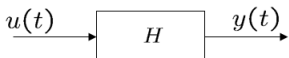
February 11, 2020

Module 5 Outline

- 1 General linear systems analysis
- 2 Responses to different test signals
- 3 First order systems & properties
- 4 Second order systems & properties
- 5 Reading sections: 5.1–5.5 Ogata, 5.1–5.4 Dorf and Bishop

What have we done so far?

- Well...So far, we know how to model a dynamical system
- + Reduce blocks to a single transfer function
- Module's goal: **analyze + characterize** input-output behavior
- Simple idea: want to know how your system is performing?
- Yes! Well, excite it with different test inputs \Rightarrow study the response



Test Inputs

- ① Impulse input: $u(t) = \delta(t)$, Output: impulse-response, $y_i(t) = \mathcal{L}^{-1}[H(s)] = h(t)$
- ② Step input: $u(t) = 1^+(t)$, Output: step-response, $y_s(t) = ?$
 - Step input characterizes system's ability to track sudden input changes
- ③ Ramp input: $u(t) = t$, Output: ramp-response, $y_r(t) = ?$
 - Ramp input characterizes system's ability to track varying input
 - Why are these important? How is this useful? Relationship between them:

unit-impulse response

$$\mathcal{L}[h(t)] = H(s)$$



$$h(t) = \frac{d}{dt}s(t)$$

unit-step response

$$\mathcal{L}[s(t)] = \frac{H(s)}{s}$$

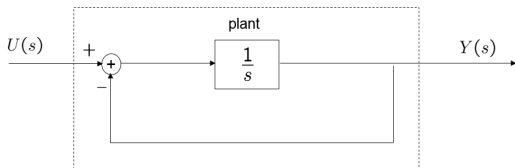


$$s(t) = \frac{d}{dt}r(t)$$

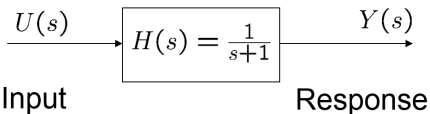
unit-ramp response

$$\mathcal{L}[r(t)] = \frac{H(s)}{s^2}$$

Example



- First, we find the overall transfer function, $H(s)$
- Solution:



$$u(t) = \delta(t)$$

$$h(t) = e^{-t} \cdot 1(t)$$

$$u(t) = 1(t)$$

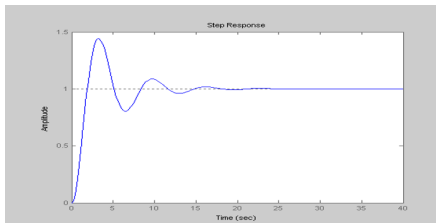
$$s(t) = (1 - e^{-t}) \cdot 1(t)$$

$$u(t) = t \cdot 1(t)$$

$$r(t) = (t - 1 + e^{-t}) \cdot 1(t)$$

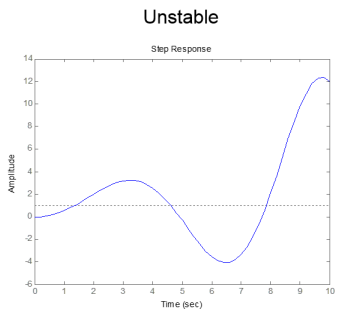
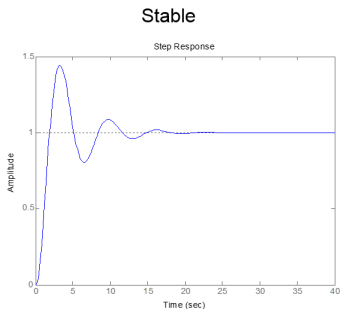
Transient Vs. Steady State Responses

- Any output for linear system is decomposed of: $y(t) = y_{ss}(t) + y_{tr}(t)$
- $y_{ss}(t)$: steady-state response — signifies the system's ability to *eventually* track input signals after few seconds
- $y_{tr}(t)$: transient response — path the output took to reach SS
- Overly oscillatory $y_{tr}(t)$ is usually bad for systems. Why?
- Slow transient response is typically undesirable. Why?
- Example:

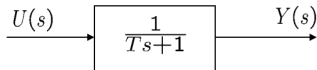
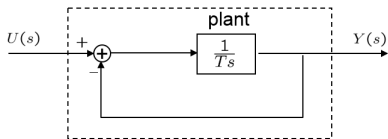


Stable Vs. Unstable Systems? How to Characterize?

- **Stable system: step response converges to a finite value OR**
- **Impulse response converges to...?**
- **Unstable system: step response output doesn't converge**
- Example:



First Order Systems



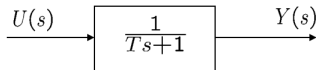
- What's the meaning of first order systems?
- They're characterized by this TF:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ts + 1}, \quad T = \text{time constant}$$

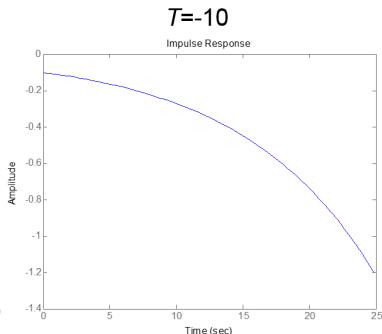
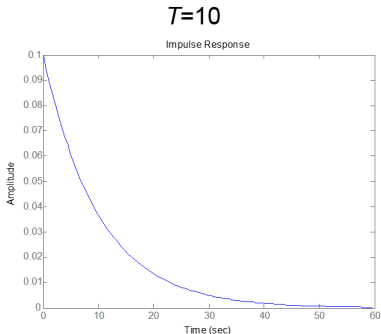
- Can we derive the ODE related to the input and output?
- What happens if $T < 0$? $T > 0$?
- What happens when T varies? For $T > 0$:
 - Larger $T \Rightarrow$ slower decay (larger time-constant)
 - Smaller $T \Rightarrow$ faster decay (smaller time-constant)

First Order System: Stability Analysis & Impulse Response

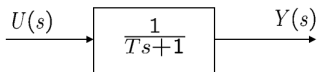
- For smaller T , system will go to zero faster
- Plots show the impulse response, $h(t)$



System is stable if $T > 0$, and unstable if $T < 0$



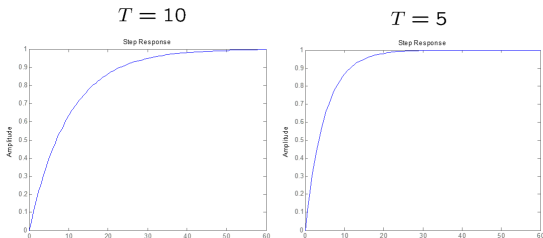
Step Response



- What is the step response of the FOS?

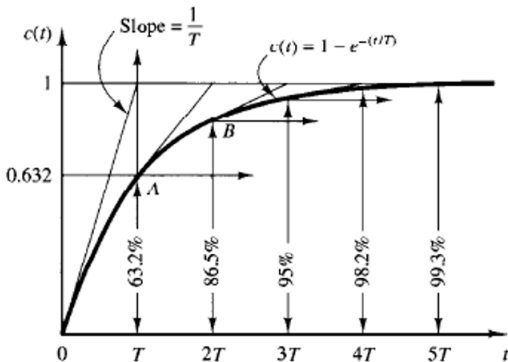
$$Y_{step}(s) = H(s)U(s) = \frac{1}{Ts + 1} \frac{1}{s} = \frac{1}{s} - \frac{1}{s + \frac{1}{T}} \Rightarrow y_{step}(t) = 1 - e^{-\frac{t}{T}}$$

- Similar to impulse response, smaller $T \Rightarrow$ faster response
- Example:

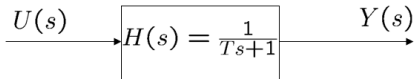


Time Constant and Step Response

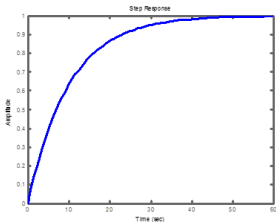
- What happens if $t = T$, i.e., $t = 1$ time constant?
- Answer: $y_{step}(t = T) = 1 - e^{-\frac{T}{T}} = 1 - e^{-1} = 0.632$
- How many time constants do we need to reach steady-state (SS)?
- Solution: after $t \geq 5T$, we reach 99.3% of SS



Effect of Poles on Step Response

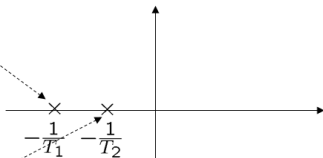
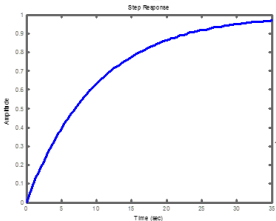


T_1



1st order system has a pole at $-1/T$
 Smaller time constant means pole is more to the left
 Smaller time constant means faster step responses

T_2

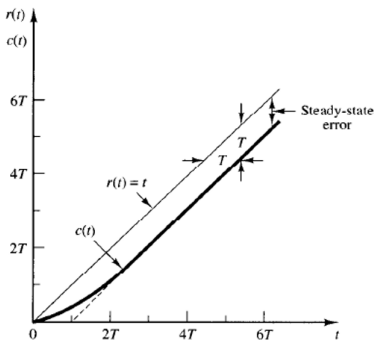


$$T_1 < T_2$$

Ramp Response of FOSs

- So far, we've done impulse and step responses of FOSs
- Now: ramp response. Again, why are we doing this?
- What is the impulse response of the FOS?

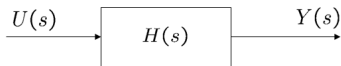
$$Y_{ramp}(s) = H(s)U(s) = \frac{1}{Ts + 1} \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} - \frac{T^2}{Ts + 1} \Rightarrow y_{ramp}(t) = t - T + Te^{-\frac{t}{T}}$$



Important Remarks on FOSs

- Location of the pole (i.e., $p = -1/T$) determines the response of FOSs
- Transient will settle down (i.e., stable) if p is in the LHP
- If the pole is further on the LHP, transients will settle down faster
- Why are there no oscillations for step response of FOSs?
- I'll give you brownie points if you guess :)

SOSs: Introduction and Definition



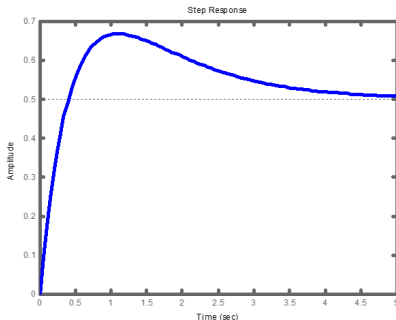
- Generic TF of SOSs:

$$H(s) = \frac{b_0s^2 + b_1s + b_2}{a_0s^2 + a_1s + a_2}$$

- Most important thing for SOSs: the location of the poles of $H(s)$
- SOS is **called stable** if **all poles are in the LHP**

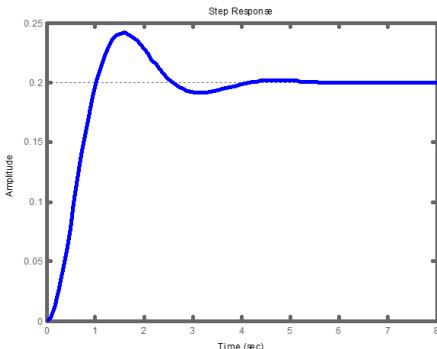
Step Response of Stable SOS

- Example: $H(s) = \frac{2s + 1}{s^2 + 3s + 2}$ — poles: $p_1 = -2, p_2 = -1$
- What's $y_{step}(t)$? We should know how to obtain that by now
- $y_{step}(t) = e^{-t} - 1.5e^{-2t} + 0.5$
- Poles p_1 and p_2 contribute to a term in $y_{step}(t)$
- However, since both poles are stable, step response converges to a SS value = 0.5 — notice the so-called *overshoot*



What happens if the poles are stable, but complex?

- Another motivating example: $H(s) = \frac{1}{s^2 + 2s + 5}$
- Poles: $p_{1,2} = -1 \pm 2i$ — stable poles (LHP), complex conjugates
- Step response: $y_{step}(t) = 0.2 - 0.2e^{-t} \cos(2t) + 0.1e^{-t} \sin(2t)$
- Sines and cosines \Rightarrow oscillations, right? What's the SS value?
- Step response:

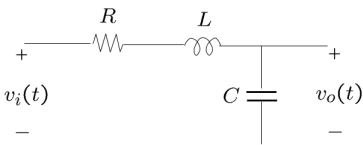


More Common Standard Form of SOSs

- The most common standard form of SOSs:

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- This form: (a) represents only one family of SOSs, (b) denominator polynomial has **+ve coefficients**, (c) $H(0) = 1$
- Definitions: (a) $\omega_n \equiv$ **undamped natural frequency**, (b) $\zeta \equiv$ **damping ratio**
- $\omega_n > 0$, $\zeta > 0$

SOS Example: finding ω_n and ζ  $v_i(t)$: input $v_o(t)$: output

- Recall this circuit example from Module 3
- What was the TF? $H(s) = \frac{1}{LCs^2 + RCs + 1}$
- This is not in the standard form (previous slide)

- In standard form: $H(s) = \frac{1/LC}{s^2 + \underbrace{R/L}_{=2\zeta\omega_n} \cdot s + \underbrace{1/LC}_{=\omega_n^2}}$

- Hence: $\omega_n = \sqrt{1/LC}$, $\zeta = \frac{R}{2\sqrt{\frac{L}{C}}}$

Poles of SOSs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Poles:

$$p_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

- SOS has two poles — how many cases to consider? Three cases:
 - **Underdamped case:** Two **complex conjugate** poles $\Rightarrow 0 < \zeta < 1$
 - **Critically damped case:** Two **identical real** poles $\Rightarrow \zeta = 1$
 - **Overdamped case:** Two **distinct real** poles $\Rightarrow \zeta > 1$

Case 1 — Underdamped System, $0 < \zeta < 1$

$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ has two complex poles

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

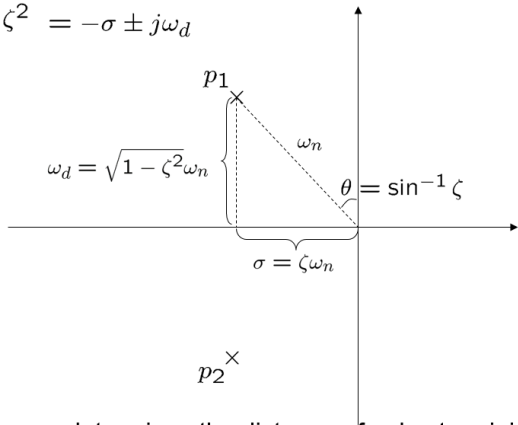
- where

$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

“damped natural frequency”

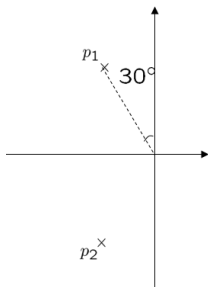
- Angle $\theta = \sin^{-1} \zeta$
- As ζ increases from 0 to 1:
 θ increases from 0 to 90 degree



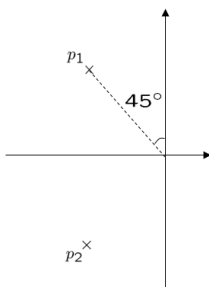
- Undamped natural frequency ω_n determines the distance of poles to origin
- Damping ratio ζ determines the angle θ

Case 1 — Underdamped System, Examples

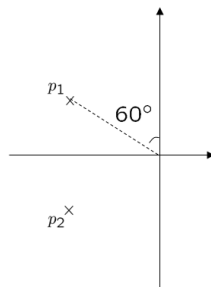
$$\zeta = 0.5$$



$$\zeta = 0.707$$



$$\zeta = 0.866$$



Case 1 — Underdamped System, Step Response

- We can easily obtain the step response given Case 1 ($0 < \zeta < 1$)
- Since we have complex poles, $p = -\sigma + j\omega_d$, taking the inverse Laplace transform for $1/(s + p)$ would yield exponentially decaying sines and cosines:

$$e^{pt} = e^{(-\sigma + j\omega_d)t} = e^{-\sigma t} (\cos(\omega_d t) + j \sin(\omega_d t))$$

- What are the transients and SS components?

Step response of $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_d^2}$

$$s(t) = \mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right]$$

$$= 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

$$\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

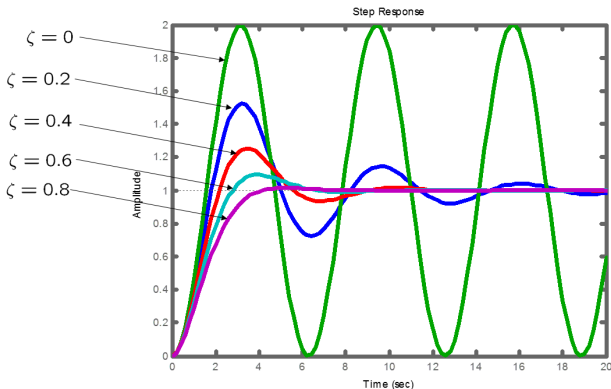
↑
steady state response

↙ ↗
transient responses

Case 1 — Underdamped System Step Response

- Here, we change ζ , while ω_n is constant for an underdamped system
- Remember that

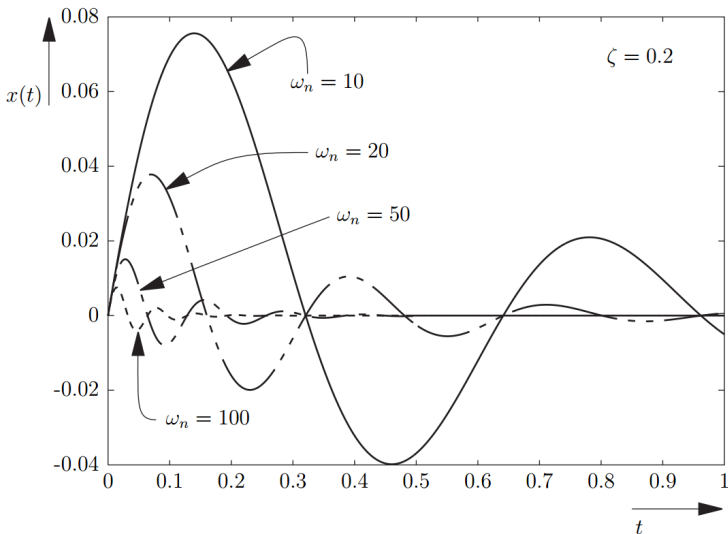
$$s(t) = y_{step}(t) = 1 - e^{-\zeta\omega_n t} \cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$



Case 1 — Underdamped System, Important Remarks

- As we saw in the previous plot for different ζ for underdamped case, we have overshoot and oscillation
- Real part of the poles ($\sigma = \zeta\omega_n$) determines transient amplitude decaying rate
- Imaginary part of the poles (ω_d) determines transient oscillation frequency
- For a given undamped system, as $\zeta \nearrow$:
 - Angle $\theta \nearrow$, poles shift more to the left, $\omega_d \searrow$
 - Overshoot \searrow
- What happens if we $\nearrow \omega_n$ and fix ζ ?

Fixing ζ and Increasing ω_n



Case 2 — Critically Damped System, $\zeta = 1$

- This case is not that interesting — not as much as Case 1
- Why? Cz we have 2 identical real poles at the same location (LHP):

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

- Poles: $p_{1,2} = -\omega_n$
- Step response? $y_{step}(t) = \mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{\omega_n^2}{(s + \omega_n)^2}\right] = 1 - e^{-\omega_n t} (1 + \omega_n t)$
- How did we get this from the step response of underdamped case?

$$y_{step}^{under}(t) = 1 - e^{-\zeta\omega_n t} \cos(\omega_d t) - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

- Well, this can be obtained by letting ζ approach 1 and use the limit of $\sin(\alpha x)/x = \alpha$ as $x \rightarrow 0$:

$$\lim_{\zeta \rightarrow 1} \frac{\sin(\omega_d t)}{\sqrt{1 - \zeta^2}} = \lim_{\zeta \rightarrow 1} \frac{\sin(\omega_n \sqrt{1 - \zeta^2} t)}{\sqrt{1 - \zeta^2}} = \omega_n t$$

Case 3 — Overdamped System, $\zeta > 1$

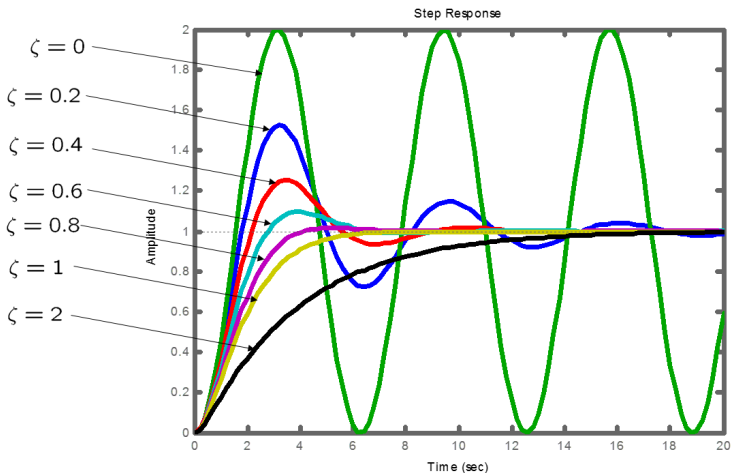
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Also, not a very interesting case...Actually, a very boring one
- Poles: **distinct real poles**, $p_{1,2} = -(\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$
- Step response:

$$y_{step}^{over}(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{p_1 t}}{p_1} - \frac{e^{p_2 t}}{p_2} \right)$$

- Can approximate overdamped second order systems as first order systems?
- Yes. How? Dominant poles...

Step Response for Different ζ



- For $\zeta \geq 1$, system response mimics what?

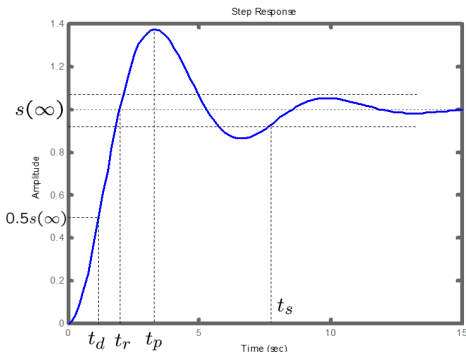
VERY Important Remarks on SOSs

- Overdamped system is slow in responding to inputs — takes time to reach SS
- That depends on how far the poles are in the LHP
- For systems without oscillations, which one responds faster to inputs? In other words, which one reaches SS faster?
 - Answer: critically damped system, $\zeta = 1$ — see previous plot
- Underdamped systems with $0.5 \leq \zeta \leq 0.8$ get close to the final value more rapidly than critically damped or overdamped system, without incurring too large overshoot
- How can we obtain impulse or ramp response of second order systems?
 - Answer: by differentiation and integration, respectively.

Time Specs of Systems

- 1 t_d : **delay time** — time for $y_{step}(t)$ to reach half of $y_{step}(\infty)$
- 2 t_r : **rise time** — time for $y_{step}(t)$ to reach first $y_{step}(\infty)$
- 3 t_p : **peak time** — time for $y_{step}(t)$ to reach first peak
- 4 M_p : **maximum overshoot** — $M_p = \frac{y_{step}(t_p) - y_{step}(\infty)}{y_{step}(\infty)}$
- 5 t_s : **settling time** — time for $y_{step}(t)$ to settle within a range of 2% 5% of $y_{step}(\infty)$

A typical step response $s(t) = y_{step}(t)$



Time Specs of Systems — 2

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Given ζ and ω_n , can we determine the time-specs in terms of them?
- I mean can we have an equations that relate the two?
- We can, yes...We'll focus on the underdamped case as three time-specs aren't defined for critically and overdamped systems
- Step response, revisited:

$$\begin{aligned} s(t) = y_{step}(t) &= 1 - e^{-\zeta\omega_n t} \cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \beta) \end{aligned}$$

- $\beta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$

Closed form Solution of Time Specs

- **Delay time:** find the smallest positive solution to $y_{step}(t_d) = 0.5$
- **Rise time:** smallest positive solution of $y_{step}(t_r) = 1 \Rightarrow$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}}{\sqrt{1 - \zeta^2} \omega_n}$$

- **Peak time:** smallest positive solution to $y'_{step}(t_p) = 0$:

$$t_p = \frac{\pi}{\omega_d}$$

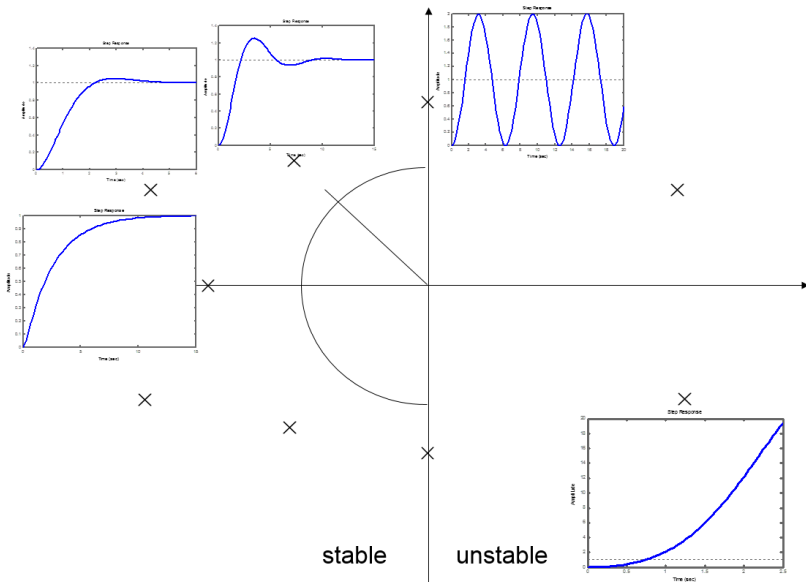
- **Maximum overshoot:**

$$M_p = \frac{y_{step}(t_p) - y_{step}(\infty)}{y_{step}(\infty)} = y_{step}(t_p) - 1 \Rightarrow$$

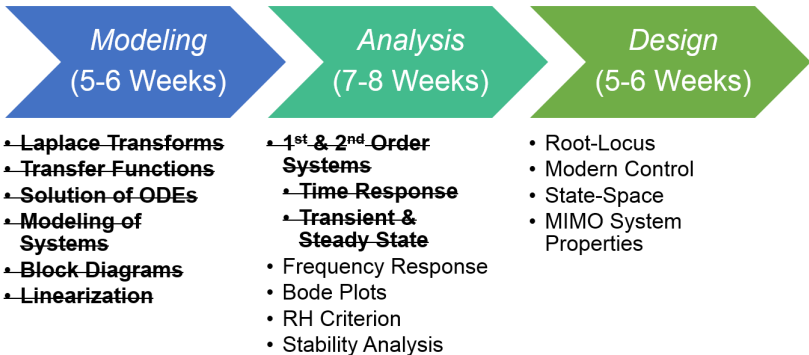
$$M_p = e^{-\frac{\zeta}{\sqrt{1 - \zeta^2}} \pi}$$

- **Settling time:** $t_s \approx \frac{4}{\zeta \omega_n}$ (2% criteria), $t_s \approx \frac{3}{\zeta \omega_n}$ (5% criteria),
- $t_p \searrow$ with ω_n ; the smaller the ζ , the larger the M_p

Effect of Pole Locations on Responses of SOSs



Where Are We Now?



Questions And Suggestions?



Thank You!

Please visit

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IFF you want to know more 😊