Module 06
Higher Order Systems, Stability Analysis & Steady-State Errors

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EE 3413: Analysis and Desgin of Control Systems

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Module 6 Outline

1. From FOSs & SOSs to higher-order systems
2. Stability of linear systems
3. Routh-Hurwitz stability criterion
4. System types & steady-state tracking errors
5. Reading sections: 5.4, 5.6, 5.8 Ogata, 5.6, 6.1, 6.2 Dorf and Bishop

Nonstandard SOSs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- So far, we analyzed the above TFs for SOSs
- What if we have a non-unit DC gain?

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - What's $y_{step}(\infty)$? Behavior won't change as much
- What if we have a zero:

$$H(s) = \frac{\alpha s\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Given an extra zero, we obtain:

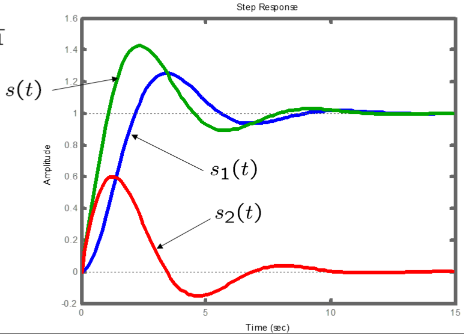
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\alpha s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = H_1(s) + H_2(s) = H_1(s) + \frac{\alpha}{\omega_n^2} sH_1(s)$$

Adding an Extra Zero

$$H(s) = H_1(s) + H_2(s) = H_1(s) + \frac{\alpha}{\omega_n^2} sH_1(s)$$

- Therefore, under any input (step, impulse, ramp), the response will be:
$$y(t) = y_1(t) + y_2(t) = y_1(t) + \frac{\alpha}{\omega_n^2} y_1'(t)$$
- $y_1(t)$: unit-step response of standard SOS; Step response example
- Zero affects overshoot in the step response

$$H(s) = \frac{s+1}{s^2+0.8s+1}$$



Higher Order Systems

- How can we analyze systems with more zeros, more poles?
- First, write the TF in this standard form:

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- Location of poles determines almost everything
- How many cases do we have?

(1) For distinct real poles:

$$H(s) = \frac{\alpha_1}{s-p_1} + \cdots + \frac{\alpha_n}{s-p_n}$$

- Unit step and impulse responses? Easy to derive

$$y_{imp}(t) = \alpha_1 e^{p_1 t} + \cdots + \alpha_n e^{p_n t}\ , y_{step}(t) = \beta_0 + \beta_1 e^{p_1 t} + \cdots + \beta_n e^{p_n t}$$

- Transients will vanish **iff** $p_1, \ldots, p_n$ are negative

Mean, Complex Poles

(2) For distinct real and complex poles:

$$H(s) = \sum_{j=1}^{q} \frac{\alpha_j}{s - p_j} + \sum_{k=1}^{r} \frac{\beta_k s + \gamma_k}{s^2 + 2\sigma_k s + \omega_k^2}$$

- You'll have to show me your PFR superpowers to obtain $\alpha_j, \beta_k, \gamma_k, \sigma_k, \omega_k \ \forall j, k$
- Unit-impulse response:

$$y_{imp}(t) = \sum_{j=1}^{q} \alpha_j e^{p_j t} + \sum_{k=1}^{r} c_k e^{-\sigma_k t} \sin(\omega_k t + \theta_k)$$

- Unit-step response:

$$y_{step}(t) = \sum_{j=1}^{q} d_j e^{p_j t} + \sum_{k=1}^{r} f_k e^{-\sigma_k t} \sin(\omega_k t + \phi_k)$$

- Similar to the previous case, transients will vanish if all poles are in the LHP

Summary & Important Remarks

- Each real pole p contributes to an exponential term in any response
- Each complex pair of poles contributes a modulated oscillation
  - The decay of these oscillations depend on whether the real-part of the pole is negative or positive
  - The magnitude of oscillations, contributions depends on residues, hence on zeros
- **Dominant poles:** poles that dominate any kind of output response
  - Dominant poles can be real (be real ok?) or complex

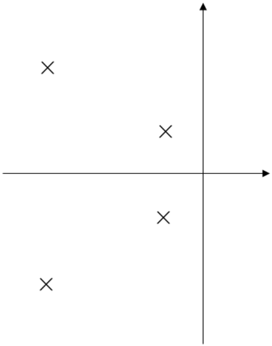
Dominant Poles — Example

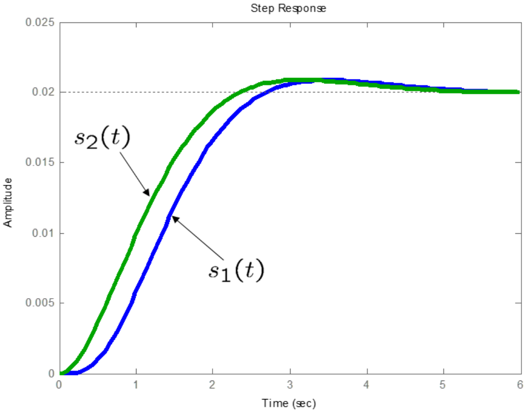
$$H_1(s) = \frac{1}{(s^2+2s+2)(s^2+8s+25)}$$

$$p_{1,2} = -1 \pm j \qquad p_{3,4} = -4 \pm j3$$

$$H_2(s) = \frac{1/25}{s^2+2s+2}$$

$$p_{1,2} = -1 \pm j$$





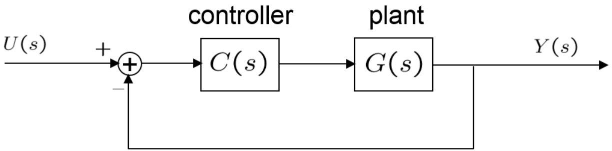
Who Likes Stability? Who Likes Instability?

- Stability: one of the most important problems in control
- *System is stable if, under bounded input, its output will converge to a finite value, i.e., transient terms will eventually vanish. Otherwise, it is unstable*
- Above definition is a tricky one—we need a quantitative one
- From now on, this system is **stable iff** all p's have **strictly negative real parts**

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- If $p_i = 0$, would the system be stable? **NO, NO.**

Design Problems Related to Stability



- **Stability Criterion**: for a given system (i.e., given $C(s), G(s)$), determine if it is stable
- **Stabilization**: for a given system that is unstable (i.e., poles of $G(s)$ are unstable), design $C(s)$ such as $\frac{Y(s)}{U(s)}$ is stable
- Most engineering design applications for control systems evolve around this simple, yet occasionally challenging idea
- Some systems **cannot be stabilized**
- For more complex $G(s)$, design of $C(s)$ is likely to be more complex
- However, this **IS NOT A RULE**

How to Infer Stability? Two Methods

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$$

- System, denoted by the above TF $H(s)$ is stable **iff**:

$$roots(a_0 s^n + a_1 s^{n-1} + \cdots + a_n = 0) \in \text{ LHP}$$

- How can we determine that? Two methods:

(1) Direct factorization, Matlab, algebra:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_n = K(s - p_1)(s - p_2) \cdots (s - p_n) = 0$$

- That cannot be done on hands (often), need a computer

(2) Routh's Stability Criterion:

- for any polynomial of any degree, *determine # of roots in the LHP, RHP, or $j\omega$ axis* without having to solve the polynomial
- Advantages: Less computations + gives discrete answers

Routh-Hurwitz Stability Criterion (RHSC)

- So, the RHSC only tells me whether a polynomial (denominator of a TF) has roots in LHP, RHP, or $j\omega$ axis, not the exact locations, which answers stability question of control systems
- The **opposite is not always true!**
- How does this work:
  - First, if $a_0s^n + a_1s^{n-1} + \cdots + a_n$ is stable, then $a_0, a_1, \cdots, a_n$ have the same sign **and** are nonzero
  - Examples: $(s^2 - s + 1)$ is not stable, $s^4 + s^3 + s^2 + 1$ is not stable
  - $s^4 + s^3 + s^2 + s + 1$ is undetermined

How to Apply the RHSC?

- **Objective:** given $a_0 s^n + a_1 s^{n-1} + \cdots + a_n \Rightarrow$ determine if polynomial is stable

(Step 1) Determine if all coefficients of $a_0 s^n + a_1 s^{n-1} + \cdots + a_n$ have the same sign & nonzero

(Step 2) If the answer to Step 1 is NO, then system is unstable

(Step 3) Arrange all the coefficients in this *Routh-Array* format:

$$\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & & & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \quad \cdots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \quad \cdots$$

RHSC Algorithm — 2

s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$					
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

(Step 4) # RHP roots = # of sign changes in the first column

(Step 5) Stability determination: $a_0 s^n + a_1 s^{n-1} + \cdots + a_n$ is stable if the first column has no sign change

RHSC Example — 1

- Determine the stability of:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

- Apply the RHSC:

$$\begin{array}{c||ccc} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & \frac{2\cdot3-4\cdot1}{2} = 1 & \frac{2\cdot5-1\cdot0}{2} = 5 & \\ s^1 & \frac{1\cdot4-2\cdot5}{1} = -6 & & \\ s^0 & =? & & \end{array}$$

(S. 4–5) # RHP roots = # of sign changes = 2 ⇒ two RHP roots ⇒ unstable polynomial

RHSC Example — 2

- What is a condition on a_0, a_1, a_2, a_3 such that the polynomial is stable (all are +ve)?

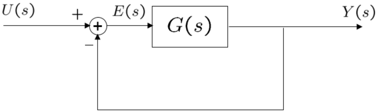
$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

- Apply the RHSC:

$$\begin{array}{c||cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 \cdot a_2 - a_0 \cdot a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

(S. 4–5) Need no sign change in the first column ⇒ need $\boxed{a_1 a_2 > a_0 a_3}$, since $a_i > 0 \, \forall i$

RHSC Example — 2



- Given the above unity-feedback system, and $G(s) = \dfrac{K}{s(s^2+10s+20)}$, find range of K s.t. the CLTF is stable
- **Solution:** first, find CLTF; $H(s) = \dfrac{K}{s^3+10s^2+20s+K}$
  - Apply the RHSC: Steps 1 and 2; $K > 0$ and:

$$
\begin{array}{c||cc}
s^3 & 1 & 20 \\
s^2 & 10 & K \\
s^1 & -\frac{1}{10}(K-200) & \\
s^0 & K &
\end{array}
$$

(S. 4–5) Need no sign change in the first column $\Rightarrow$ need $K < 200$ and $K > 0$, $\Rightarrow$ $\boxed{0 < 200 < K}$

Special Case 1

- Sign of 0? What if 1 of the entries in the first column is 0?
- **Solution**: replace 0 with ϵ, where ϵ is a small +ve number
- **Case 1:** if the sign of the coefficient above the zero (ϵ) is the same as the sign under ϵ $\Rightarrow$ there are pair of complex roots
  - **Example**: $s^3 + 2s^2 + s + 2 = 0$

$$\begin{array}{c||cc} s^3 & 1 & 1 \\ s^2 & \boxed{2} & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & \boxed{2} & \end{array}$$

- **Case 2**: if the sign of the coefficients above and below ϵ change $\Rightarrow$ there is a sign change $\Rightarrow$ apply Step 5
  - **Example**: $s^3 - 3s + 2 = (s-1)^2(s+2) = 0$

$$\begin{array}{c||cc} s^3 & \boxed{1} & -3 \\ s^2 & 0 \approx \epsilon & 2 \\ s^1 & \boxed{-3 - \dfrac{2}{\epsilon}} & \\ s^0 & 2 & \end{array}$$

Special Case 2 + Example

- What if an entire row is zero? Then we have:
  - (a) two real roots with equal magnitudes and opposite signs and/or (b) two complex conjugate roots
- Solution illustrated with this example:
  - **Example**: $p(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 = 0$

s^5	1	11	28	
s^4	5	23	12	
s^3	6.4	25.6		
s^2	3	12		
~~s^1~~	~~0~~	~~0~~		old row, define aux. polynomial : $P(s) = 3s^2 + 12$
s^1	6	0		new row, define aux. polynomial : $P'(s) = 6s + 0$
s^0	12			

(Step 4) Find roots of auxiliary polynomial: $3s^2 + 12 = 0 \Rightarrow p_{1,2} = \pm j2$

(Step 5) $p_{1,2}$ are both roots for the original polynomial

(Step 6) Count sign changes: none, hence no additional RHP roots

Another Example

- **Example**: $p(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

s^5	1	24	-25	
s^4	2	48	-50	
~~s^3~~	~~0~~	~~0~~		old row, define aux. polynomial : $P(s) = 2s^4 + 48s^2 - 50$
s^3	8	96		new row, define aux. polynomial : $P'(s) = 8s^3 + 96$
s^2	24	-50		
s^1	112.7	0		
s^0	-50			

(Step 4) Find roots of auxiliary polynomial:
$2s^4 + 48s^2 - 50 = 0 \Rightarrow p_{1,2,3,4} = \pm j5, \pm 1$

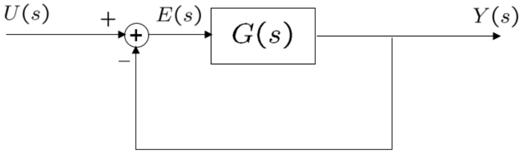
(Step 5) p_3 in RHP, then at least one RHP pole

(Step 6) Count sign changes: **once**, hence one more additional RHP root

- **Conclusion:** one RHP pole — verification:
$p(s) = (s+1)(s-1)(s+j5)(s-j5)(s+2) = 0$

Tracking Error

- What is tracking? Why is tracking important?
  - Tracking is an important task in control systems
    * Objective: track a certain reference signal ($reference(t)$ or $u(t)$)
- Often, $ref.(t)$ is a step function or piecewise constant signals
- Tracking is typically achieved via unity-feedback control systems
  - **Definition 1:** tracking error $= e(t) = u(t) - y(t)$
  - **Definition 2:** stead-state error (SSE) $= e_{ss} = e(\infty)$
- Wait, we can apply FVT here $\Rightarrow$ $\boxed{e_{ss} = \lim_{s \to 0} sE(s)}$
- **Important point**: SSE only defined if system is stable
- **Target:** study SSE for a unity-feedback system



What Inputs Can We Consider?

Unit step input: $u(t) = 1, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s}$

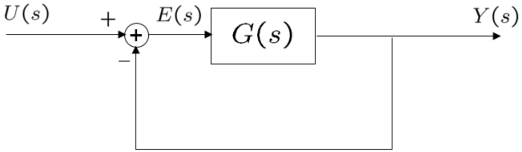
Unit ramp input: $u(t) = t, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s^2}$

Unit acceleration input: $u(t) = \frac{t^2}{2}, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s^3}$

In general: $u(t) = \frac{t^k}{k!}, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s^{k+1}}$

- Many system inputs can be approximated with scaled polynomials
- How can we do that? `polyfit` on MATLAB: http://www.mathworks.com/help/matlab/ref/polyfit.html
- If your system can track high order inputs (e.g., $u(t) = t^{10} + 5t^4 - 7$), then your system has an excellent ability in tracking *arbitrary inputs*

System Type (More Definitions)



- A **unity-feedback** system with an OLTF

$$G(s) = \frac{K(T_a s + 1)\cdots(T_m s + 1)}{s^N(T_b s + 1)\ldots(T_n s + 1)}$$

  is called type N **where N is the # of poles of** $G(s)$ **at** $s = 0$

- Examples
- **Goal**: find SSE for different **system types** & **test inputs** (unit step, impulse, ramp)

SSE for a Unit-Step Input

$$\boxed{e_{ss} = \lim_{s \to 0} sE(s), \ \textit{\textbf{if system is stable}}}$$

- We now want to find e_{ss} for any given $G(s)$
- Recall (from Module 04 and Exam I) that $\dfrac{E(s)}{U(s)} = \dfrac{1}{1+G(s)}$
- Then, what's $e_{ss} = e(\infty)$ if $u(t) = 1$?
- **Answer:** $e_{ss} = \dfrac{1}{1+K_p}$, $K_p = \lim_{s \to 0} G(s)$
- K_p is called the static position error constant
- What would e_{ss} for Type 0 systems? Type 1?
- **Answer:** Type 0, it's constant (above), Types 1 and above, it's 0
- **Conclusion 1:** Type 0 systems track unit step with finite SSE
- **Conclusion 2:** Type 1 or higher systems track unit step with 0 SSE

SSE for a Unit-Step Input

$$e_{ss} = \lim_{s \to 0} sE(s) \quad , \quad \frac{E(s)}{U(s)} = \frac{1}{1+G(s)}$$

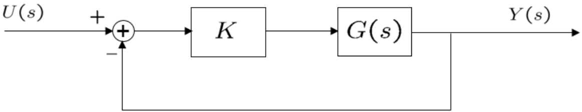
- Then, what's $e_{ss} = e(\infty)$ if $u(t) = t$?
- **Answer:** $e_{ss} = \dfrac{1}{K_v}$, $K_v = \lim_{s \to 0} sG(s)$
- K_v is called the static velocity error constant
- What would e_{ss} for Type 0 systems? Type 1?
- **Answer:** Type 0, it's infinity! Why?
- **Conclusion 1:** Type 0 systems **cannot track unit ramp input**
- **Conclusion 2:** Type 1 systems track unit ramp step with finite SSE
- **Conclusion 3:** Type 2 or higher systems track unit ramp unit step with 0 SSE

Summary of the Results

	Unit step input u(t)=1	Unit ramp input u(t)=t	Acceleration input u(t)=t²/2
Type 0 systems	$\frac{1}{1+K_p}$ $K_p = G(0)$	∞	∞
Type 1 systems	0	$\frac{1}{K_v}$ $K_v = \lim_{s\to 0} sG(s)$	∞
Type 2 systems	0	0	$\frac{1}{K_a}$ $K_a = \lim_{s\to 0} s^2G(s)$

- You should not memorize any of these results — you should be able to derive all of these 9 results
- Before you compute anything, verify that the system is stable

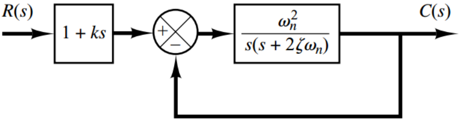
Design Example 1



$$G(s) = \frac{1}{s(s^2+s+1)(s+2)}$$

- For the above given system, and assuming that $u(t) = 1$, find K such that the SSE is as small as possible
- **Answer:**

Design Example 2



- Assume that $u(t) = t$, find K such that the SSE is zero
- **Answer:** First, find the overall transfer function:

$$H(s) = \frac{C(s)}{R(s)} = (1+ks)\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

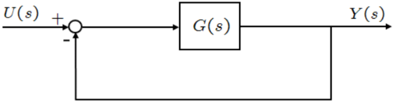
- Now, find $E(s)$ then e_{ss} via FVT

$$E(s) = R(s) - C(s) = \left(\frac{s^2+2\zeta\omega_n s-\omega_n^2 ks}{s^2+2\zeta\omega_n s+\omega_n^2}\right)R(s) = \left(\frac{s^2+2\zeta\omega_n s-\omega_n^2 ks}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\frac{1}{s^2}$$

$$\Rightarrow e_{ss} = e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left(\frac{s^2+2\zeta\omega_n s-\omega_n^2 ks}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\frac{1}{s^2} = \frac{2\zeta\omega_n-\omega_n^2 k}{\omega_n^2}$$

We want $e_{ss} = 0 \Rightarrow$ *set* $k = \frac{2\zeta}{\omega_n}$ *to achieve that*

Design Example 3



- For the above given system, and assuming that

$$G(s) = \frac{K}{s^3 + s^2 + 2s - 4},$$

obtain the SSE for unit step input when $K = 1, 5,$ **or** 10.

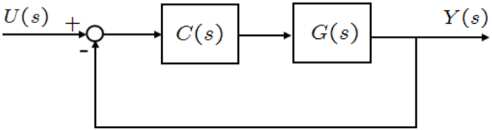
(1) First, we have to find the range for K s.t. system (CLTF) is stable

(2) Routh-Array for $s^3 + s^2 + 2s + K - 4 = 0$:

$$\begin{array}{c||cc} s^3 & 1 & 2 \\ s^2 & 1 & K-4 \\ s^1 & 6-K & \\ s^0 & K-4 & \end{array} \Rightarrow \text{system is stable if } \boxed{4 < K < 6}$$

(3) $\therefore$ for $K = 1, 10$, SSE doesn't exist. System is Type 0 $\Rightarrow$ for $K = 5$, SSE is: $e_{ss} = \dfrac{1}{1 + G(0)} = -4$

Design Example 4



- For the above given system, assume that

$$G(s) = \frac{1}{s^3 + s^2 + 2s - 0.5}, \quad C(s) = 1 + \frac{K}{s}.$$

  For $K \geq 0$, obtain the range of K such that the CLTF is stable
- Do this problem at home
- **Solution:** $0 < K < 0.75$

Course Progress

Modeling (5-6 Weeks)

- ~~Laplace Transforms~~
- ~~Transfer Functions~~
- ~~Solution of ODEs~~
- ~~Modeling of Systems~~
- ~~Block Diagrams~~
- ~~Linearization~~

Analysis (7-8 Weeks)

- ~~1st & 2nd Order Systems~~
  - ~~Time Response~~
  - ~~Transient & Steady State~~
- Frequency Response
- Bode Plots
- ~~RH Criterion~~
- ~~Stability Analysis~~

Design (5-6 Weeks)

- Root-Locus
- Modern Control
- State-Space
- MIMO System Properties

Questions And Suggestions?

Thank You!

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IFF you want to know more ☺