

# Module 07

## Control Systems Design & Analysis via Root-Locus Method

**Ahmad F. Taha**

**EE 3413: Analysis and Design of Control Systems**

*Email:* [ahmad.taha@utsa.edu](mailto:ahmad.taha@utsa.edu)

*Webpage:* <http://engineering.utsa.edu/ataha>



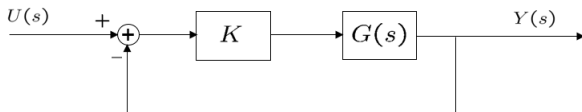
March 19, 2020

# Module 7 Outline

- ① Introduction to root-locus
- ② Root-locus plots and examples
- ③ Root-locus steps and rules
- ④ Design of feedback systems via root-locus
- ⑤ More examples
- ⑥ Reading sections: 6.1–6.5 Ogata, 7.1–7.4 Dorf and Bishop

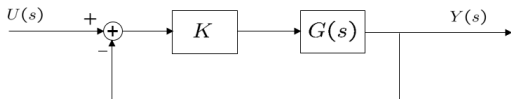
# Introduction to Root-Locus (RL)

- Previously, we learned that CLTF poles determine everything
- Poles determine stability, transients, steady-state error
- What if we change one of the constants?
- If we change one parameter, how will the CL poles change?
- Will the system go unstable? Will the transients change?
- **Example:** remember the finding the  $K$  example in Module 06?
- Finding the best  $K$  for this system...We'll learn how CLTF change if we change  $K$



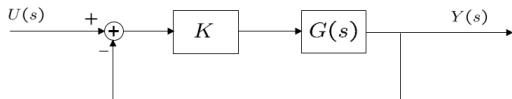
$$G(s) = \frac{1}{s(s^2+s+1)(s+2)}$$

# RL Preliminaries

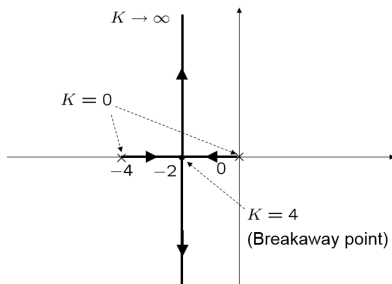


- **Root-Locus:** graph showing how poles vary as  $K \nearrow$  from  $0 \rightarrow \infty$
- We'll start with the above system (unity feedback), but RL is general for different systems
- RL: a graph showing closed-loop poles as 1 system parameter changes
- **Definition:** the characteristic polynomial (CP) of CLTF is  $1 + KG(s)H(s) = 0$
- This equation determines the RL
- For the above system,  $H(s) = 1$
- RL method can be generalized for different style of systems

# RL Example

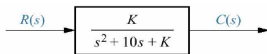


- Example: let's plot the RL for  $G(s) = 1/(s(s+4))$
- CP:  $1 + K \frac{1}{s(s+4)} = 0 \Rightarrow s^2 + 4s + K = 0 \Rightarrow p_{1,2} = -2 \pm \sqrt{4-K}$
- For  $K = 0$ :  $p_{1,2} = 0, -4$ ;  $K = 4$ :  $p_{1,2} = -2, -2$ ;  $K > 4$ :  $p_{1,2} =$  complex #



# RL Another Example

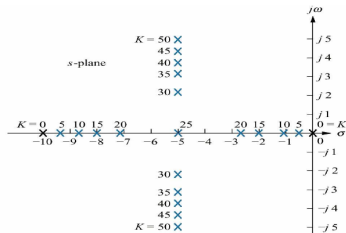
- We can always obtain the RL by taking different values for parameter  $K$  and then evaluating the corresponding CLTF poles (roots of CP)
- Another example:



The location of poles as a function of  $K$  can be calculated as

The corresponding root locus can be drawn

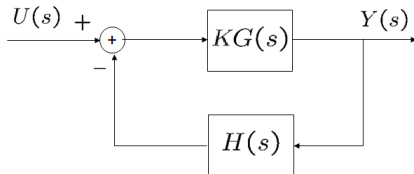
$K$	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



# RL Plot Intro

- The previous examples are relatively easy
- Roots of CP easy to derive in terms of  $K$  — not the case for 3rd and higher order systems
- **Objective:** have a systematic way of drawing the RL for **any system**
- RL has many branches, is a weird-looking plot (which makes sense)
- RL helps infer design values for gains of control systems
- Tedious solution: have a table for different  $K$ s and corresponding CLTF poles (as in the previous example)
- Then, draw the root locus by connecting the points
- We'll learn a more systematic approach that's easy to execute

# The Rules



**Defs.** OLTF =  $G(s)H(s)$ ;  $n_p, n_z$  = number of poles, zeros of OLTF

**Rule 1** RL is symmetric with respect to **the real-axis**—remember that

**Rule 2** RL has  $n$  branches,  $n = n_p$

**Rule 3** Mark poles ( $n_p$ ) and zeros ( $n_z$ ) of  $G(s)H(s)$  with 'x' and 'o'

**Rule 4** Each branch starts at OLTF poles ( $K = 0$ ), ends at OLTF zeros or at infinity ( $K = \infty$ )

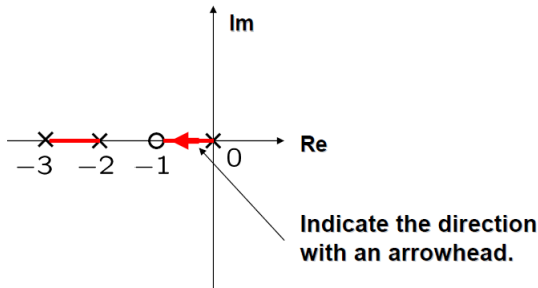
**Rule 5** RL has branches on x-axis. These branches exist on real axis portions where the **total # of poles + zeros** to the right is an odd #

## Example 2, Executing Rules 1–5

$$G(s) = \frac{s + 1}{s(s + 2)(s + 3)}, H(s) = 1 \Rightarrow \text{OLTF} = K \frac{s + 1}{s(s + 2)(s + 3)}$$

- Rules 1–5:

- $n_p = 3, n_z = 1, \#$  of Branches = 3, plot so far:



- Hence, we have to figure out the shape **of only 1 more branch**

# RL Rules 6 and 7: Angles of Asymptotes

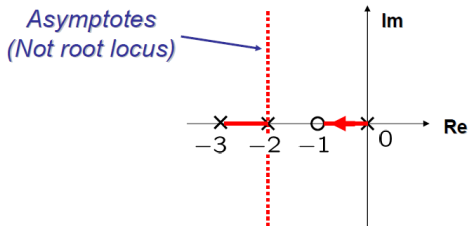
**Rule 6** Asymptotes angles: RL branches ending at OL zeros at  $\infty$  approach the asymptotic lines with angles:

$$\phi_q = \frac{(1 + 2q)180}{n_p - n_z} \text{ deg}, \forall q = 0, 1, 2, \dots, n_p - n_z - 1$$

**Rule 7** Real-axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} \text{Re}(p_i) - \sum_{j=1}^{n_z} \text{Re}(z_j)}{n_p - n_z}$$

- Solution of Rules 6 and 7:
- $\phi_q = 90$  and  $270$  degrees,  $\sigma_A = \frac{0+(-2)+(-3)-(-1)}{2} = -2$



# Rule 8: Break-away & Break-in Points

$$1 + KG(s)H(s) = 0 \Rightarrow K(s) = \frac{-1}{G(s)H(s)} \Rightarrow \frac{dK(s)}{ds} = -\frac{d}{ds} \left[ \frac{1}{G(s)H(s)} \right]$$

**Rule 8-1** RL branches intersect the real-axis at points where  $K$  is at an extremum for real values of  $s$ . We find the breakaway points by finding solutions (i.e.,  $s^*$  solutions) to:

$$\frac{dK(s)}{ds} = 0 = -\frac{d}{ds} \left[ \frac{1}{G(s)H(s)} \right] = 0 \Rightarrow \frac{d}{ds} [G(s)H(s)] = 0 \Rightarrow \text{obtain } s^*$$

**Rule 8-2** After finding  $s^*$  solutions (you can have a few), check whether the corresponding  $K(s^*) = \frac{-1}{G(s^*)H(s^*)} = K^*$  is **real positive #**

**Rule 8-3 Breakaway pt.:**  $K_{max}^*$  (-ve  $K''(s^*)$ ), **Break-in pt:**  $K_{min}^*$  (+ve  $K''(s^*)$ )

# Solution for Rule 8

- Recall that  $K(s) = \frac{-1}{G(s)H(s)}$

- For the given  $G(s), H(s)$ , we get:

$$\frac{d}{ds} [G(s)H(s)] = 0 \Rightarrow$$

- Recall derivative rule for fractions (quotient rule)

- Hence:

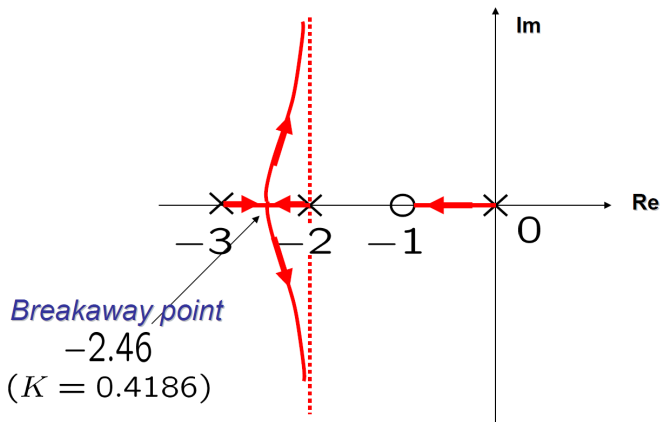
$$\frac{d}{ds} [G(s)H(s)] = 0 \Rightarrow \frac{d}{ds} \left( \frac{s+1}{s(s+2)(s+3)} \right) = \frac{-2s^3 - 8s^2 - 10s - 6}{(s(s+2)(s+3))^2} = 0$$

Rule 8-1  $s^* = -2.46, -0.76 \pm 0.79i$  (3 solutions, since numerator is 3rd degree polynomial, now we check which  $s^*$  yields  $K > 0$ , next)

Rule 8-2  $K(s^*) = \frac{-1}{G(s^*)H(s^*)} = 0.418, 1.79 \pm 4.27 \Rightarrow K^* = 0.418$

Rule 8-3  $K''(s^* = -2.46) < 0 \Rightarrow$  breakaway point

# Solution for Rule 8



## Two More Rules (None Apply in This Example)

**Rule 9 Angle of Departure (AoD):** defined as the angle from a complex pole or Angle of Arrival (AoA) at a complex zero:

$$\text{AoD from a complex pole : } \phi_p = 180 - \sum_i \angle p_i + \sum_j \angle z_j$$

$$\text{AoA at a complex zero : } \phi_z = 180 + \sum_i \angle p_i - \sum_j \angle z_j$$

- $\sum_i \angle p_i$  is the sum of all angles of vectors to a complex pole in question from other poles,  $\sum_j \angle z_j$  is the sum of all angles of vectors to a complex pole in question from other zeros
- '∠' denotes the angle of a complex number

**Rule 10** Determine whether the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i$$

and finding the  $\omega$  and  $K$  that solves the above equation. The value of  $\omega$  you get is the frequency at which the RL crosses the imaginary y-axis and the  $K$  you get is the associated gain for the controller

# Example 3 — RL for $G(s) = \frac{1}{s(s+1)(s+5)}$ , $H(s) = 1$

Defs.  $n_p = 3$ ,  $n_z = 0$ , OLTF =  $\frac{1}{s(s+1)(s+5)}$

Rule 1 RL is symmetric with respect to the real-axis—OK!

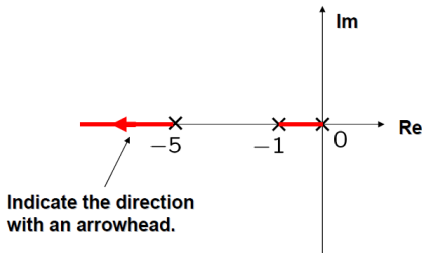
Rule 2 RL has 3 branches

Rule 3 Marking poles (no zeros)

Rule 4 Branches start @ OLTF p's ( $K=0$ ), ends @ OLTF z's or @  $K = \infty$

Rule 5 Draw branches on the x-axis via the odd total number of z's and p's

Solution ... so far:



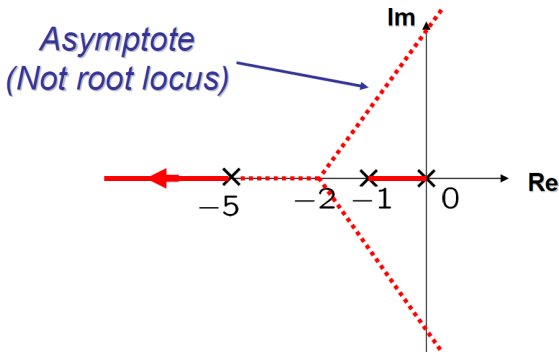
## Example 3 (Cont'd)

Rule 6 Asymptotes angles:

$$\phi_q = \frac{(1 + 2q)180}{3} \quad \forall q = 0, 1, 2 \quad || = \{60, 180, 300\} \text{ deg}$$

Rule 7 Real axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} \text{Re}(p_i) - \sum_{j=1}^{n_z} \text{Re}(z_j)}{n_p - n_z} = -2$$



## Example 3 (Cont'd)

Rule 8-1 Find breakaway/breaking points:

$$\frac{dK(s)}{ds} = 0 = -\frac{d}{ds} \left[ \frac{1}{G(s)H(s)} \right] = 0 \Rightarrow \frac{d}{ds} [G(s)H(s)] = 0 \Rightarrow \text{obtain } s^*$$

$$\Rightarrow 3s^2 + 12s + 5 = 0 \Rightarrow s_{1,2}^* = -2 \pm \frac{\sqrt{21}}{3} = -0.47, -3.3$$

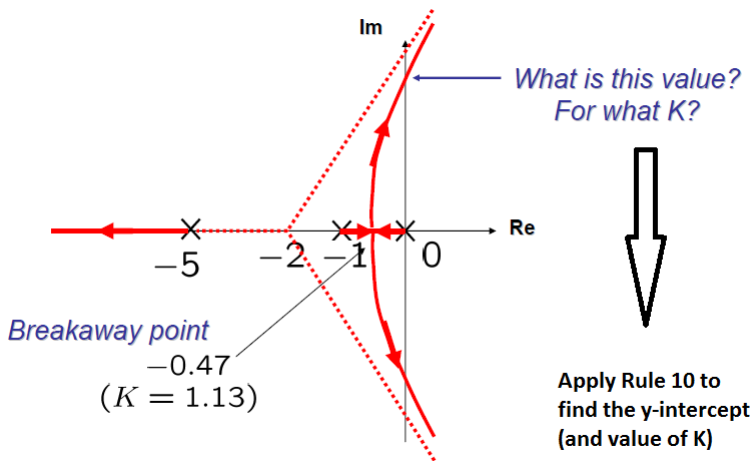
– Notice that  $s_2^*$  does not belong to the RL

Rule 8-2 Find corresponding values for  $K^*$ :

$$K_1^* = 1.13 (s_1^* \approx -0.47), K_2^* = -13.1 (s_2^* \approx -3.3) \Rightarrow \boxed{K^* = 1.13} > 0$$

Rule 8-3  $s_1^*$  is a **breakaway point** since  $K''(s_1^*) < 0$

# Solution for Rule 8



# Rules 9 and 10

**Rule 9** Undefined since we do not have complex poles or zeroes

**Rule 10** Determine where the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i$$

$$\Rightarrow 1 + \frac{K}{j\omega(j\omega + 1)(j\omega + 5)} = 0 + 0j$$

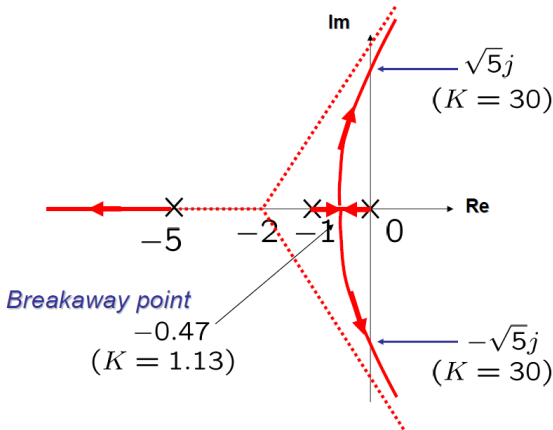
$$\Rightarrow -6\omega^2 + K + j(-\omega^3 + 5\omega) = 0 + 0j$$

$$\Rightarrow K = 6\omega^2, \quad \omega(-\omega^2 + 5) = 0$$

$$\Rightarrow \boxed{\omega^* = \pm j\sqrt{5}, K^* = 30} \text{ — the y-axis crossings}$$

# Overall Solution

Always remember that the RL plot of any system is the way poles change for different values of  $K$  or any other parameter. Hence, for a stable system, we would want  $0 < K < 30$ . We could have checked that via the Routh array too.



# Example 4 — RL for $G(s) = \frac{s + 2}{s^2 + 2s + 3}$ , $H(s) = 1$

Defs.  $n_p = 2$ ,  $n_z = 1$ , OLTF =  $\frac{s + 2}{s^2 + 2s + 3}$

P's, Z's  $p_{1,2} = -1 \pm j\sqrt{2}$ ,  $z_1 = -2$

Rule 1 RL is symmetric with respect to the real-axis—OK!

Rule 2 RL has **only** 2 branches

Rule 3 Marking poles and zeros

Rule 4 Branches start @ OLTF p's ( $K=0$ ), ends @ OLTF z's or @  $K = \infty$

Rule 5 Draw branches on the x-axis via the odd total number of z's and p's

Solution ... So far

## Example 4 (Cont'd)

Rule 6 Asymptotes angles:

$$\phi_q = \frac{(1 + 2q)180}{1} \quad \forall q = 0 \quad || = 180 \text{ deg}$$

Rule 7 Real axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} \text{Re}(p_i) - \sum_{j=1}^{n_z} \text{Re}(z_j)}{n_p - n_z} = 0 \quad \text{— no actual intersection}$$

Solution

## Example 4 (Cont'd)

$$K(s) = -\frac{s^2 + 2s + 3}{s + 2}$$

Rule 8-1 Find breakaway/breaking points:

$$\frac{dK(s)}{ds} = 0 \implies \text{obtain } s^*$$

$$\implies s^2 + 4s + 1 = 0 \implies s_{1,2}^* = -3.73, -0.26$$

- Notice that  $s_2^*$  does not belong to the RL so most likely it's not a solution

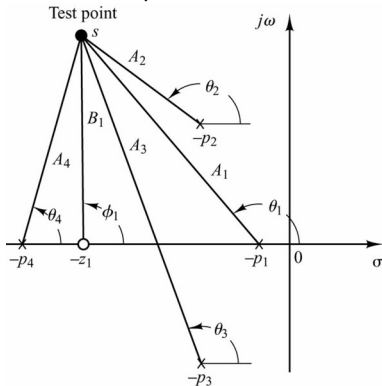
Rule 8-2 Find corresponding values for  $K^*$ :

$$K_1^* = 5.46 \quad (s_1^* = -3.73), \quad K_2^* = -1.46, \quad (s_2^* = -0.26) \implies \boxed{K^* = 5.46} > 0$$

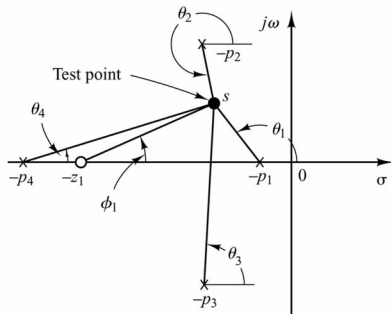
Rule 8-3  $s_1^*$  is a **break-in point** since  $K''(s_1^*) > 0$

# Rule 9

Figures show angle measurements from open-loop poles & open-loop zero to test point  $s$



(a)



(b)

**Rule 9** Angle of departure from  $p_1 = 180 - 90 + 55 = 145$ , from  $p_2 = 180 - 270 + 245 = -145$ , no angle of arrival since we have no complex zeros

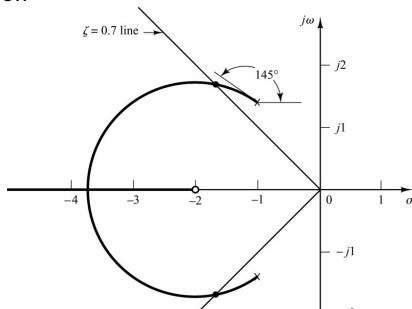
# Rule 10

**Rule 10** Determine where the RL crosses the imaginary y-axis by setting:

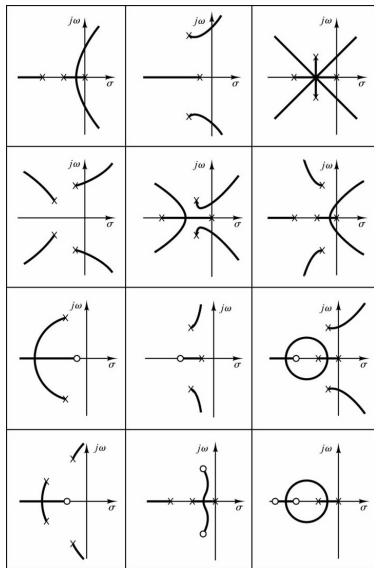
$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i \Rightarrow 1 + \frac{K(j\omega + 2)}{(j\omega)^2 + 2j\omega + 3} = 0 + 0j$$

$$\Rightarrow \omega^2 - 3 + j(-2\omega - k\omega - 2k) = 0 + 0j \Rightarrow \omega = \pm\sqrt{3}$$

- This value for  $\omega$  cannot be on the root locus, hence no crossings on the y-axis
- Overall solution



# Generic RL Plots



Example 5 — RL for  $G(s) = \frac{1}{s(s^2 + 4s + 5)}$ ,  $H(s) = 1$

Defs.  $n_p = 3$ ,  $n_z = 0$ , OLTF =  $\frac{1}{s(s^2 + 4s + 5)}$

Rule 1 RL is symmetric with respect to the real-axis—OK!

Rule 2 RL has 3 branches

Rule 3 Marking poles (no zeros)

Rule 4 Branches start @ OLTF p's (K=0), ends @ OLTF z's or @  $\infty$

Rule 5 Draw branches on the x-axis via the odd total number of z's and p's

Solution ... so far

# Example 5 (Cont'd)

Rule 6 Asymptotes angles:

$$\phi_q = \frac{(1 + 2q)180}{3} \quad \forall q = 0, 1, 2 \quad || = \{60, 180, 300\} \text{ deg}$$

Rule 7 Real axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} \text{Re}(p_i) - \sum_{j=1}^{n_z} \text{Re}(z_j)}{n_p - n_z} = -\frac{0 + 2 + 2}{3} = -1.333$$

## Example 5 (Cont'd)

Rule 8-1 Find breakaway/breaking points:

$$K(s) = -(s^3 + 4s^2 + 5s) \quad dK(s)/ds = 0 \Rightarrow s_{1,2}^* = -1, -1.667$$

- Note that both solutions are on the root locus

Rule 8-2 Find corresponding values for  $K^*$ :

$$K_{1,2}^* = 2, 1.85 > 0$$

Rule 8-3  $s_1^*$  is a breakaway point,  $s_2^*$  is a break-in point (second derivative rule of  $K(s)$ )

Rule 9 Angle of departure from the complex poles:

$$\phi_{p_1} = 190 - 90 - 153.43 = -63.43 \text{ deg}, \phi_{p_2} = 63.43 \text{ deg}$$

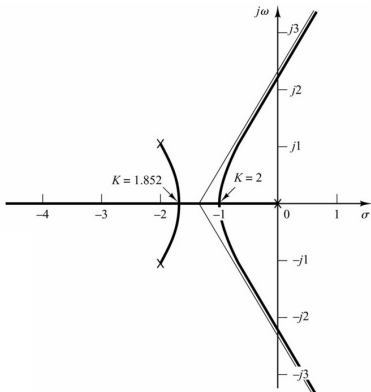
# Rule 10 + Overall Solution

**Rule 10** Determine where the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i \Rightarrow (j\omega)^3 + 4(j\omega)^2 + 5(j\omega) + K = 0 + 0j$$

$$\Rightarrow \boxed{\omega^* = \pm\sqrt{5}, K^* = 20} \text{ — the y-axis crossings}$$

Solution



# RL Module Summary

- Root-Locus: a graphical representation of the CLTF poles (can tell stability from that) as a function of one design parameter
- This design parameter — in this Module — was the gain  $K$
- RL gives more information about the poles in comparison with other stability criterion
- RHSC: gives only ranges for  $K$  for stable systems
- RL: gives a plot of how the poles are changing as a function of  $K$
- You have to practice a lot of RL plots if you wanna master the RL arts

# Questions And Suggestions?



**Thank You!**

Please visit

[engineering.utsa.edu/ataha](http://engineering.utsa.edu/ataha)

**IFF** you want to know more 😊