# Final Value Theorem

$$x(\infty) = \lim_{s\to 0} sX(s)$$

Proof: From the derivative property

$$sX(s) = \int_{0^-}^{\infty} x'(t)e^{-st}dt + x(0^-).$$

$$\begin{aligned}
\lim_{s\to 0} sX(s) &= \lim_{s\to 0}\left(\int_{0^-}^{\infty} x'(t)e^{-st}dt\right) + x(0^-) \\
&= \lim_{s\to 0}\left(\int_{0^-}^{\infty} x'(t)dt\right) + x(0^-) \\
&= [x(\infty) - x(0^-)] + x(0^-) \\
&= x(\infty)
\end{aligned}$$

Conditions: 1) all poles have non-negative real parts. 2) at most one pole at origin.

# Initial Value Theorem

$$\boxed{x(0^+) = \lim_{s\to\infty} sX(s)} \qquad \text{if } \lim_{s\to\infty} X(s) = 0$$

Proof: From the derivative property

$$sX(s) = \int_{0^-}^{\infty} x'(t)e^{-st}dt + x(0^-).$$

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\left(\int_{0^-}^{\infty} x'(t)e^{-st}dt\right) + x(0^-)$$

$$= \lim_{s\to\infty}\left(\int_{0^-}^{0^+} x'(t)e^{0}dt + \int_{0^+}^{\infty} x'(t)e^{-st}dt\right) + x(0^-)$$

$$= \lim_{s\to\infty}\left(x(t)\Big|_{0^-}^{0^+} + 0\right) + x(0^-)$$

$$= x(0^+)$$

# More Generally

If $X(s)$ can be written as

$$X(s) = \tilde{X}(s) + \sum_{n=0}^{\infty} a_n s^n$$

where $\tilde{X}(s)$ converges to zero as $s \to \infty$. In this case,

$$\lim_{s \to \infty} s\tilde{X}(s) = x(0^+).$$

# Reference

Initial Conditions, Generalized Functions, and the Laplace Transform. Troubles at the origin.
Kent H. Lundberg, Haynes R. Miller, and David L. Trumper

www-math.mit.edu/~hrm/papers/lmt.pdf