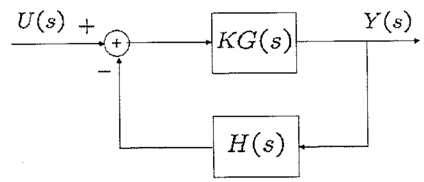
Root-Locus, Rules 1–5



Defs. OLTF $= G(s)H(s)$; n_p, n_z = number of poles, zeros of OLTF

Defs. Characteristic Polynomial (CP) $= 1 + KG(s)H(s)$

$$\Rightarrow \boxed{1 + KG(s)H(s) = 0 \Rightarrow K(s) = \frac{-1}{G(s)H(s)}}$$

Rule 1 RL is always symmetric with respect to **the real-axis**—remember that

Rule 2 RL has n branches, $n = n_p$

Rule 3 Mark poles (n_p) and zeros (n_z) of $G(s)H(s)$ with 'x' and 'o'

Rule 4 Each branch starts at OLTF poles ($K = 0$), ends at OLTF zeros or at infinity ($K = \infty$)

Rule 5 RL has branches on x-axis. These branches exist on real axis portions where the **total # of poles + zeros** to the right is an odd #

Root-Locus, Rules 9–10

Rule 9 **Angle of Departure (AoD):** defined as the angle from a complex pole or Angle of Arrival (AoA) at a complex zero:

$$\text{AoD from a complex pole}: \ \phi_p = 180 - \sum_i \angle p_i + \sum_j \angle z_j$$

$$\text{AoA at a complex zero}: \ \phi_z = 180 + \sum_i \angle p_i - \sum_j \angle z_j$$

- $\sum_i \angle p_i$ is the sum of all angles of vectors to a complex pole in question from other poles, $\sum_j \angle z_j$ is the sum of all angles of vectors to a complex pole in question from other zeros
- '$\angle$' denotes the angle of a complex number

Rule 10 Determine whether the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i$$

and finding the ω and K that solves the above equation. The value of ω you get is the frequency at which the RL crosses the imaginary y-axis and the K you get is the associated gain for the controller. You should obtain two equations (real = 0 and imaginary = 0) with two unknowns (K, ω). From there, you solve for K, ω pairs

Root-Locus, Rules 6–8

Rule 6 Asymptotes angles: RL branches ending at OL zeros at ∞ approach the asymptotic lines with angles:

$$\phi_q = \frac{(1+2q)180}{n_p - n_z} \text{ deg}, \forall q = 0, 1, 2, \ldots, n_p - n_z - 1$$

Rule 7 Real-axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} Re(p_i) - \sum_{j=1}^{n_z} Re(z_j)}{n_p - n_z}$$

Rule 8-1 RL branches intersect the real-axis at points where K is at an extremum for real values of s. Remember that:

$$\boxed{1 + KG(s)H(s) = 0 \Rightarrow K(s) = \frac{-1}{G(s)H(s)}}$$

We find the breakaway points by finding solutions (i.e., s^* solutions) to:

$$\frac{dK(s)}{ds} = 0 = -\frac{d}{ds}\left[\frac{1}{G(s)H(s)}\right] = 0 \Rightarrow \frac{d}{ds}[G(s)H(s)] = 0 \Rightarrow \text{obtain } s^*$$

Rule 8-2 After finding s^* solutions (you can have a few), check whether the corresponding $K(s^*) = \dfrac{-1}{G(s^*)H(s^*)} = K^*$ is **real positive #**

Rule 8-3 **Breakaway pt.:** K^*_{max} (-ve $K''(s^*)$), **Break-in pt:** K^*_{min} (+ve $K''(s^*)$)

Lead Compensator Design Algorithm

- **Objective:** design $G_c^{ld}(s) = K\frac{s+z}{s+p}$, such that ζ_d, ω_{nd} are given
- To find K, z, p, follow this algorithm:

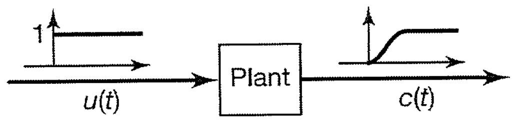
0. Find s_d for $s_d^2 + 2\zeta_d\omega_{nd}s_d + \omega_{nd}^2 = 0, s_d = ...$
1. Find *angle of deficiency* ϕ, as follows:

$$\theta = \angle G(s_d) \Rightarrow \phi = -180 - (\theta)$$

2. Connect s_d to the origin
3. Draw a horizontal line to the left from s_d
4. Find the bisector of the above two lines
5. Draw 2 lines that make angles $\phi/2$ & $-\phi/2$ with the bisector
6. Their intersections with the real lines are $-p$ and $-z$
7. Find K:

$$1 + KG(s)G_c^{ld}(s) = 0 \Rightarrow \boxed{1 = |KG(s)G_c^{ld}(s)|} \quad \text{here } G_c^{ld}(s) \text{ is without the K}$$

Ziegler-Nichols Rule: First Method



Step 1 Obtain plant's unit step response experimentally[1]

- Unit step response is S-shaped for many plants
- Only valid if the step-response is S-shaped

Step 2 Obtain delay time L from the experimental plot

Step 3 Obtain time constant T from the experimental plot

Step 4 Use tuning rule table to determine K_p, T_i, T_d given L, T (next slide)

[1] In industrial applications, control engineers usually specify the performance of the controlled system based on the system step response.

Obtaining K_p, T_i, T_d via Tuning Method 1

$$\boxed{G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)}$$

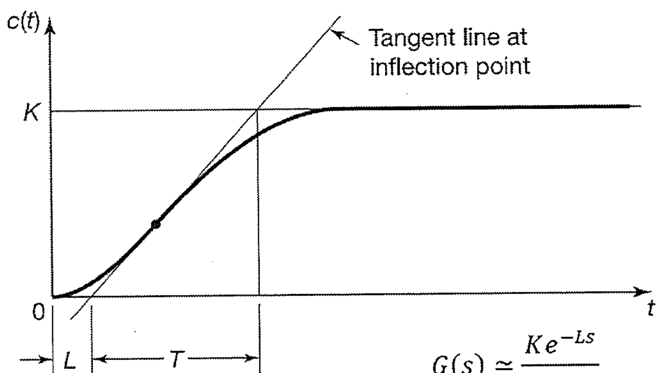
Type of Controller	K_p	T_i	T_d
P	$\frac{T}{KL}$	∞	0
PI	$\frac{0.9T}{KL}$	$3.3L$	0
PID	$1.2\frac{T}{KL}$	$2L$	$0.5L$

- If you want PID control, choose the 3rd row & compute the parameters:

$$G_{PID}(s) = G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = 0.6T\frac{\left(s + \frac{1}{L}\right)^2}{s}$$

- These rules give only the starting point for the design—you might get a better response if you obtain a different set of constants

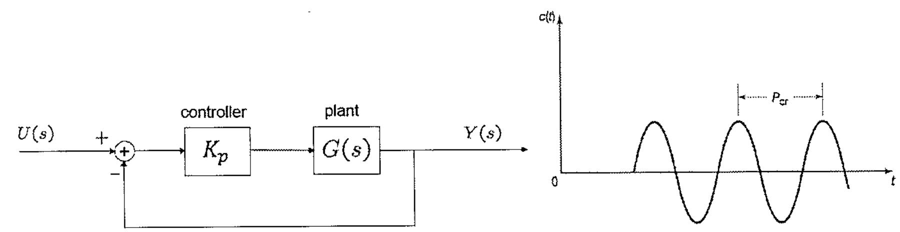
Obtaining L, T from Experimental Plot



$$G(s) \simeq \frac{Ke^{-Ls}}{Ts + 1}$$

- Of course, this is an approximation, but you have to be accurate with your computation of L and T

Ziegler-Nichols Rule: Second Method



Step 1 Increase K_p until step response of the closed-loop system **has sustained oscillations**

- If no oscillation occurs for all values of K_p, this method is not applicable

Step 2 Record K_{cr} (critical value of gain K_p) and P_{cr} (period of the oscillation); see above right figure

Step 3 Use tuning rule table to determine K_p, T_i, T_d given K_{cr}, P_{cr} (next slide)

Obtaining K_p, T_i, T_d via Tuning Method 2

$$\boxed{G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)}$$

Type of Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$P_{cr}/1.2$	0
PID	$0.6K_{cr}$	$P_{cr}/2$	$P_{cr}/8$

$$G_{PID}(s) = G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = 0.075 K_{cr} P_{cr} \frac{\left(s + \frac{4}{P_{cr}}\right)^2}{s}$$

You should also know how to compute K_{cr} via the Routh Array analytically rather than experimentally, and then compute P_{cr}.

Observable Canonical Form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

TF to Controllable Canonical Form

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}}$$

Controllable Canonical form:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}_{Ax(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)}_{Bu(t)}$$

$$y(t) = \underbrace{\begin{bmatrix} b_n - a_n b_0 | & b_{n-1} - a_{n-1} b_0 | & \cdots | & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}_{Cx(t)} + \underbrace{b_0 u(t)}_{Du(t)}$$

Diagonal Canonical Form:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

State-space to TF:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t) \\ \frac{Y(s)}{U(s)} &= \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \end{aligned}$$

Response of LTI Systems

- MIMO (or SISO) LTI dynamical system:

$$\begin{aligned} \dot{x}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \boldsymbol{x}(t_0) = \boldsymbol{x}_{t_0} = \text{given} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

- The to the above ODE is given by:

$$\boxed{\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau}$$

- Clearly the output solution is:

$$\boxed{\boldsymbol{y}(t) = \underbrace{\boldsymbol{C}\left(e^{A(t-t_0)}\boldsymbol{x}_{t_0}\right)}_{\text{zero input response}} + \underbrace{\boldsymbol{C}\left(\int_{t_0}^{t} e^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau\right) + \boldsymbol{D}\boldsymbol{u}(t)}_{\text{zero state response}}}$$

Controllability and Observability

Controllability Test

For a system with n states and m control inputs (i.e., $\boldsymbol{B} \in \mathbb{R}^{n\times m}$), the test for controllability is that matrix

$$\mathcal{C} = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix} \in \mathbb{R}^{n\times nm}$$

has full row rank (i.e., $\text{rank}(\mathcal{C}) = n$).

Observability Test

For a system with n states and p outputs (i.e., $\boldsymbol{C} \in \mathbb{R}^{p\times n}$), the test for observability is that matrix $\mathcal{O} = \begin{bmatrix}\boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1}\end{bmatrix} \in \mathbb{R}^{np\times n}$ has full column rank (i.e., $\text{rank}(\mathcal{C}) = n$).