Practice Problems 10, Laplace Transforms

Evaluate the following Laplace transforms.

1. $\mathcal{L}(e^{-5t}) = \frac{1}{s-(-5)} = \frac{1}{s+5}$

2. $\mathcal{L}(t^2e^{2t})$

   Since $\mathcal{L}(t^2) = \frac{2}{s^3}$,

   $\mathcal{L}(t^2e^{2t}) = \frac{2}{(s-2)^3}$

3. $\mathcal{L}(\sin(t)e^{-4t})$

   Since $\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$,

   Since $\mathcal{L}(\sin(t)e^{-4t}) = \frac{1}{(s+4)^2+1}$

4. $\mathcal{L}(\cos(5t)e^{7t}) = \frac{(s-7)}{(s-7)^2+5^2}$

Evaluate the following Inverse Laplace transforms.

5. $\mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\right)$ has a denominator that can be factored into linear terms:

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} \text{ by partial fractions} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\text{so } \mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) = e^{-t} - e^{-2t}$$

6. $\mathcal{L}^{-1}\left(\frac{s+5}{(s+2)^3}\right)$ This expression is almost in the form $\frac{1}{s^3}$ but it needs to be reorganized. In particular we need to cancel out the s in the numerator.

$$\frac{s+5}{(s+2)^3} \quad \text{add 2, substract 2 from numerator}$$

$$= \frac{(s+\mathbf{2})-\mathbf{2}+5}{(s+2)^3} \quad \text{separate (s+2) term out}$$

$$= \frac{s+2}{(s+2)^3} + \frac{3}{(s+2)^3} \quad \text{cancel, then put in form for table}$$

$$= \frac{1}{(s+2)^2} + 3\left(\frac{1}{\mathbf{2}}\right) \cdot \underbrace{\frac{\mathbf{2}}{(s+2)^3}}_{\text{shifted version of } \mathcal{L}(t^2)}$$

$$\text{so } \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2} + \frac{3}{2}\frac{2}{(s+2)^3}\right)$$

$$= te^{-2t} + \frac{3}{2}t^2e^{-2t}$$

7. $\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+5}\right)$ This expression cannot be factored into linear terms so we complete the squared to get a shifted sine or cosine form.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+5}\right) &= \mathcal{L}^{-1}\left(\frac{s}{(s^2+2s\mathbf{+1-1}+5}\right)\\ &= \mathcal{L}^{-1}\left(\frac{s}{(s+1)^2+4}\right)\end{aligned}$$

This is not *quite* the cosine form: we need an $(s+1)$ factor on the top, because *every* s must be in $(s+1)$ form to use the rule $\mathcal{L}^{-1}(F(s-a)) = f(t)e^{at}$.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{(s\mathbf{+1})\mathbf{-1}}{(s+1)^2+4}\right)\\ &= \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+4}\right)\end{aligned}$$

Now we just need a 2 in the numerator of the second term to get a shifted sine transform.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - \frac{\mathbf{1}}{\mathbf{2}}\mathcal{L}^{-1}\left(\frac{\mathbf{2}}{(s+1)^2+4}\right)\\ &= \cos(2t)e^{-t} - \frac{1}{2}\sin(2t)e^{-t}\end{aligned}$$

8. $\mathcal{L}^{-1}\left(\frac{s+1}{s^2-6s+13}\right)$ Like the last problem, this denominator cannot be factored into linear terms, so we complete the square to get a shifted sine or cosine form. See the example above for the relevant comments along the way: the solution approach is identical.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s+1}{s^2-6s+13}\right) &= \mathcal{L}^{-1}\left(\frac{s+1}{(s^2-6s\mathbf{+9})\mathbf{-9}+13}\right)\\ &= \mathcal{L}^{-1}\left(\frac{s+1}{((s-3)^2+4}\right)\\ &= \mathcal{L}^{-1}\left(\frac{(s+\mathbf{-3})\mathbf{+3}+1}{(s-3)^2+4}\right)\\ &= \mathcal{L}^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) + \frac{4}{\mathbf{2}}\mathcal{L}^{-1}\left(\frac{\mathbf{2}}{(s-3)^2+4}\right)\\ &= \cos(2t)e^{3t} + 2\sin(2t)e^{3t}\end{aligned}$$

Solve the following initial value problems using Laplace transforms.

9. $x'' + 3x' + 2x = 0,\ x(0) = 0,\ x'(0) = -2$

Taking Laplace of both sides,

$$\underbrace{\left[s^2X(s) - s\cdot 0 - (-2)\right]}_{\mathcal{L}(x'')} + 3\underbrace{\left[sX(s) - 0\right]}_{\mathcal{L}(x')} + 2X(s) = 0$$

$$\text{Group } X(s) \text{ terms on left:} \qquad (s^2+3s+2)X(s) = -2$$

$$X(s) = \frac{-2}{s^2+3s+2}$$

We now try to put the RHS into a form matching the table entries. Since the denominator can be factored into linear terms, we do that and then use partial fractions to separate the factors.

$$\begin{aligned} X(s) &= \frac{-2}{(s+1)(s+2)} \\ &= \frac{A}{s+1} + \frac{B}{s+2} \\ \text{Solving for } A \text{ and } B \text{ gives} & \\ &= \frac{2}{s+2} - \frac{2}{s+1} \\ \text{so, taking inverse Laplace of both sides,} & \\ \mathcal{L}^{-1}(X(s)) &= \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{2}{s+1}\right) \\ x(t) &= 2e^{-2t} - 2e^{-t} \end{aligned}$$

Check: this satisfies $x(0) = 0$, and (differentiating) that $x'(0) = -4 + 2 = -2$.
Both functions, e^{-2t} and e^{-t}, also satisfy the original DE, so this solution satisfies both the equation and the initial conditions given.

10. $x' + 2x = -4,\ x(0) = 0$
Taking Laplace of both sides,

$$\underbrace{\left[sX(s) - 0\right]}_{\mathcal{L}(x')} + 2X(s) = \frac{-4}{s}$$

$$\text{Group } X(s) \text{ terms on left:} \qquad (s+2)X(s) = \frac{-4}{s}$$

$$X(s) = \frac{-4}{s(s+2)}$$

We now try to put the RHS into a form matching the table entries. Since the denominator can be factored into linear terms, we do that and then use partial fractions to separate the factors.

$$X(s) = \frac{-4}{s(s+2)}$$
$$= \frac{A}{s} + \frac{B}{s+2}$$

Solving for A and B gives

$$= \frac{2}{s+2} - \frac{2}{s}$$

so, taking inverse Laplace of both sides,

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{2}{s}\right)$$
$$x(t) = 2e^{-2t} - 2$$

11. $y'' + 4y = 1,\ y(0) = 0,\ y'(0) = 0$

Taking Laplace of both sides,

$$\underbrace{\left[s^2 Y(s) - s \cdot 0 - 0\right]}_{\mathcal{L}(y'')} + 4Y(s) = 1$$

Group $Y(s)$ terms on left: $\quad (s^2+4)Y(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s(s^2+4)}$$

We now try to put the RHS into a form matching the table entries. Since the denominator can be factored into s and s^2+4, we do that and then use partial fractions to separate the factors.

$$Y(s) = \frac{1}{s(s^2+4)}$$
$$= \frac{A}{s} + \frac{Bs + C}{s^2+4}$$

Solving for A, B and C gives

$$= \frac{1/4}{s} - \frac{(1/4)s + 0}{s^2+4}$$
$$= \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$$

so, taking inverse Laplace of both sides,

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right)$$
$$y(t) = \frac{1}{4} - \frac{1}{4}\cos(2t)$$

12. $y'' - 2y' = -4,\ y(0) = 0,\ y'(0) = 0$

Taking Laplace of both sides,

$$\underbrace{\left[s^2Y(s) - s\cdot 0 - 0\right]}_{\mathcal{L}(y'')} - 2\underbrace{\left[sY(s) - 0\right]}_{\mathcal{L}(y')} = \frac{-4}{s}$$

Group $Y(s)$ terms on left: $(s^2 - 2s)Y(s) = \frac{-4}{s}$

$$\begin{aligned} Y(s) &= \frac{-4}{s(s^2-2s)} = \frac{-4}{s^2(s-2)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} \text{ Solving for } A,\ B \text{ and } C \text{ gives} \\ &= \frac{1}{s} + \frac{2}{s^2} - \frac{1}{s-2} \end{aligned}$$

so, taking inverse Laplace of both sides,

$$\begin{aligned} \mathcal{L}^{-1}\left(Y(s)\right) &= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{2}{s^2} - \frac{1}{s-2}\right) \\ y(t) &= 1 + 2t - e^{2t} \end{aligned}$$

Note: if we were to have solved this problem using the y_c and y_p approach, we would have a case where $y_c = c_1$ (constant solution), so our assumed form for y_p would have needed a t multiplier to avoid the overlap. Using Laplace transforms avoids the need for this special-case logic when you are building the solution.

13. $x' - 3x = 39\sin(2t),\ x(0) = 2$

Taking Laplace of both sides,

$$\underbrace{\left[sX(s) - 2\right]}_{\mathcal{L}(x')} - 3X(s) = \frac{39}{s^2+4}$$

Group $X(s)$ terms on left: $(s-3)X(s) = 2 + \frac{39}{s^2+4}$

$$X(s) = \frac{2}{s-3} + \frac{39}{(s-3)(s^2+4)}$$

Looking just at the more complicated right-hand term, $\frac{39}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4}$

Solving for A, B and C gives

$$= \frac{3}{s-3} + \frac{-3s-9}{s^2+4}$$

Combining with the other term gives

$$\begin{aligned} X(s) &= \left(\frac{2}{s-3}\right) + \left(\frac{3}{s-3} + \frac{-3s-9}{s^2+4}\right) \\ &= \frac{5}{s-3} - 3\frac{s}{s^2+4} - \frac{9}{\mathbf{2}}\frac{\mathbf{2}}{s^2+4} \\ \text{so } x(t) &= 5e^{3t} - 3\cos(2t) - \frac{9}{2}\sin(2t) \end{aligned}$$

14. $x'' + 6x' + 34x = -34,\ x(0) = 2,\ x'(0) = 3$

Taking Laplace of both sides,

$$\underbrace{\left[s^2X(s) - s\cdot 2 - (3)\right]}_{\mathcal{L}(x'')} + 6\underbrace{\left[sX(s) - 0\right]}_{\mathcal{L}(x')} + 34X(s) = \frac{-34}{s}$$

Group $X(s)$ terms on left: $\quad (s^2 + 6s + 34)X(s) = 2s + 3 - \frac{34}{s}$

$$X(s) = \frac{2s+3}{s^2+6s+34} - \frac{34}{s(s^2+6s+34)}$$

Focusing on the complicated right-hand term,

$$\frac{-34}{s(s^2+6s+34)} = \frac{A}{s} + \frac{Bs+C}{s^2+6s+34}$$

Solving for A and B gives

$$= \frac{-1}{s} + \frac{s+6}{s^2+6s+34}$$

so, taking inverse Laplace of both sides,

$$\begin{aligned} \mathcal{L}^{-1}\left(X(s)\right) &= \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{2}{s+1}\right) \\ x(t) &= 2e^{-2t} - 2e^{-t} \end{aligned}$$