

## Practice Problems 10, Laplace Transforms

Evaluate the following Laplace transforms.

1.  $\mathcal{L}(e^{-5t}) = \frac{1}{s - (-5)} = \frac{1}{s + 5}$

2.  $\mathcal{L}(t^2 e^{2t})$

Since  $\mathcal{L}(t^2) = \frac{2}{s^3}$ ,

$$\mathcal{L}(t^2 e^{2t}) = \frac{2}{(s - 2)^3}$$

3.  $\mathcal{L}(\sin(t)e^{-4t})$

Since  $\mathcal{L}(\sin(t)) = \frac{1}{s^2 + 1}$ ,

Since  $\mathcal{L}(\sin(t)e^{-4t}) = \frac{1}{(s + 4)^2 + 1}$

4.  $\mathcal{L}(\cos(5t)e^{7t}) = \frac{(s - 7)}{(s - 7)^2 + 5^2}$

Evaluate the following Inverse Laplace transforms.

5.  $\mathcal{L}^{-1}\left(\frac{1}{s^2 + 3s + 2}\right)$  has a denominator that can be factored into linear terms:

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 1)(s + 2)} \text{ by partial fractions} = \frac{1}{s + 1} - \frac{1}{s + 2}$$

$$\text{so } \mathcal{L}^{-1}\left(\frac{1}{s^2 + 3s + 2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s + 1} - \frac{1}{s + 2}\right) = e^{-t} - e^{-2t}$$

6.  $\mathcal{L}^{-1}\left(\frac{s + 5}{(s + 2)^3}\right)$  This expression is almost in the form  $\frac{1}{s^3}$  but it needs to be reorganized. In particular we need to cancel out the  $s$  in the numerator.

$$\frac{s + 5}{(s + 2)^3} \quad \text{add 2, subtract 2 from numerator}$$

$$= \frac{(s + 2) - 2 + 5}{(s + 2)^3} \quad \text{separate (s+2) term out}$$

$$= \frac{s + 2}{(s + 2)^3} + \frac{3}{(s + 2)^3} \quad \text{cancel, then put in form for table}$$

$$= \frac{1}{(s + 2)^2} + 3\left(\frac{1}{2}\right) \cdot \underbrace{\frac{2}{(s + 2)^3}}$$

shifted version of  $\mathcal{L}(t^2)$

$$\text{so } \mathcal{L}^{-1}\left(\frac{1}{(s + 2)^2} + \frac{3}{2} \frac{2}{(s + 2)^3}\right)$$

$$= te^{-2t} + \frac{3}{2} t^2 e^{-2t}$$

7.  $\mathcal{L}^{-1}\left(\frac{s}{s^2 + 2s + 5}\right)$  This expression cannot be factored into linear terms so we complete the squared to get a shifted sine or cosine form.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{s^2 + 2s + 5}\right) &= \mathcal{L}^{-1}\left(\frac{s}{(s^2 + 2s + \mathbf{1} - \mathbf{1} + 5)}\right) \\ &= \mathcal{L}^{-1}\left(\frac{s}{(s + 1)^2 + 4}\right)\end{aligned}$$

This is not *quite* the cosine form: we need an  $(s + 1)$  factor on the top, because *every*  $s$  must be in  $(s + 1)$  form to use the rule  $\mathcal{L}^{-1}(F(s - a)) = f(t)e^{at}$ .

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{(s + 1)^2 + 4}\right) &= \mathcal{L}^{-1}\left(\frac{(s + \mathbf{1}) - \mathbf{1}}{(s + 1)^2 + 4}\right) \\ &= \mathcal{L}^{-1}\left(\frac{s + 1}{(s + 1)^2 + 4}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s + 1)^2 + 4}\right)\end{aligned}$$

Now we just need a 2 in the numerator of the second term to get a shifted sine transform.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s + 1}{(s + 1)^2 + 4}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s + 1)^2 + 4}\right) &= \mathcal{L}^{-1}\left(\frac{s + 1}{(s + 1)^2 + 4}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{\mathbf{2}}{(s + 1)^2 + 4}\right) \\ &= \cos(2t)e^{-t} - \frac{1}{2}\sin(2t)e^{-t}\end{aligned}$$

8.  $\mathcal{L}^{-1}\left(\frac{s + 1}{s^2 - 6s + 13}\right)$  Like the last problem, this denominator cannot be factored into linear terms, so we complete the square to get a shifted sine or cosine form. See the example above for the relevant comments along the way: the solution approach is identical.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s + 1}{s^2 - 6s + 13}\right) &= \mathcal{L}^{-1}\left(\frac{s + 1}{(s^2 - 6s + \mathbf{9}) - \mathbf{9} + 13}\right) \\ &= \mathcal{L}^{-1}\left(\frac{s + 1}{((s - 3)^2 + 4)}\right) \\ &= \mathcal{L}^{-1}\left(\frac{(s + \mathbf{-3}) + \mathbf{3} + 1}{(s - 3)^2 + 4}\right) \\ &= \mathcal{L}^{-1}\left(\frac{s - 3}{(s - 3)^2 + 4}\right) + \frac{4}{2}\mathcal{L}^{-1}\left(\frac{\mathbf{2}}{(s - 3)^2 + 4}\right) \\ &= \cos(2t)e^{3t} + 2\sin(2t)e^{3t}\end{aligned}$$

Solve the following initial value problems using Laplace transforms.

9.  $x'' + 3x' + 2x = 0$ ,  $x(0) = 0$ ,  $x'(0) = -2$

Taking Laplace of both sides,

$$\underbrace{[s^2 X(s) - s \cdot 0 - (-2)]}_{\mathcal{L}(x'')} + 3 \underbrace{[sX(s) - 0]}_{\mathcal{L}(x')} + 2X(s) = 0$$

Group  $X(s)$  terms on left:  $(s^2 + 3s + 2)X(s) = -2$

$$X(s) = \frac{-2}{s^2 + 3s + 2}$$

We now try to put the RHS into a form matching the table entries. Since the denominator can be factored into linear terms, we do that and then use partial fractions to separate the factors.

$$X(s) = \frac{-2}{(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

Solving for  $A$  and  $B$  gives

$$= \frac{2}{s+2} - \frac{2}{s+1}$$

so, taking inverse Laplace of both sides,

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{2}{s+1}\right)$$

$$x(t) = 2e^{-2t} - 2e^{-t}$$

Check: this satisfies  $x(0) = 0$ , and (differentiating) that  $x'(0) = -4 + 2 = -2$ . Both functions,  $e^{-2t}$  and  $e^{-t}$ , also satisfy the original DE, so this solution satisfies both the equation and the initial conditions given.

10.  $x' + 2x = -4$ ,  $x(0) = 0$

Taking Laplace of both sides,

$$\underbrace{[sX(s) - 0]}_{\mathcal{L}(x')} + 2X(s) = \frac{-4}{s}$$

Group  $X(s)$  terms on left:  $(s+2)X(s) = \frac{-4}{s}$

$$X(s) = \frac{-4}{s(s+2)}$$

We now try to put the RHS into a form matching the table entries. Since the denominator can be factored into linear terms, we do that and then use partial fractions to separate the

factors.

$$\begin{aligned} X(s) &= \frac{-4}{s(s+2)} \\ &= \frac{A}{s} + \frac{B}{s+2} \end{aligned}$$

Solving for  $A$  and  $B$  gives

$$= \frac{2}{s+2} - \frac{2}{s}$$

so, taking inverse Laplace of both sides,

$$\begin{aligned} \mathcal{L}^{-1}(X(s)) &= \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{2}{s}\right) \\ x(t) &= 2e^{-2t} - 2 \end{aligned}$$

11.  $y'' + 4y = 1$ ,  $y(0) = 0$ ,  $y'(0) = 0$

Taking Laplace of both sides,

$$\underbrace{[s^2Y(s) - s \cdot 0 - 0]}_{\mathcal{L}(y'')} + 4Y(s) = 1$$

Group  $Y(s)$  terms on left:  $(s^2 + 4)Y(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s(s^2 + 4)}$$

We now try to put the RHS into a form matching the table entries. Since the denominator can be factored into  $s$  and  $s^2 + 4$ , we do that and then use partial fractions to separate the factors.

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2 + 4)} \\ &= \frac{A}{s} + \frac{Bs + C}{s^2 + 4} \end{aligned}$$

Solving for  $A$ ,  $B$  and  $C$  gives

$$\begin{aligned} &= \frac{1/4}{s} - \frac{(1/4)s + 0}{s^2 + 4} \\ &= \frac{1}{4} \frac{1}{s} - \frac{1}{4} \frac{s}{s^2 + 4} \end{aligned}$$

so, taking inverse Laplace of both sides,

$$\begin{aligned} \mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(\frac{1}{4} \frac{1}{s} - \frac{1}{4} \frac{s}{s^2 + 4}\right) \\ y(t) &= \frac{1}{4} - \frac{1}{4} \cos(2t) \end{aligned}$$

12.  $y'' - 2y' = -4$ ,  $y(0) = 0$ ,  $y'(0) = 0$

Taking Laplace of both sides,

$$\underbrace{[s^2Y(s) - s \cdot 0 - 0]}_{\mathcal{L}(y'')} - 2 \underbrace{[sY(s) - 0]}_{\mathcal{L}(y')} = \frac{-4}{s}$$

Group  $Y(s)$  terms on left:  $(s^2 - 2s)Y(s) = \frac{-4}{s}$

$$\begin{aligned} Y(s) &= \frac{-4}{s(s^2 - 2s)} = \frac{-4}{s^2(s - 2)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 2} \text{ Solving for } A, B \text{ and } C \text{ gives} \\ &= \frac{1}{s} + \frac{2}{s^2} - \frac{1}{s - 2} \end{aligned}$$

so, taking inverse Laplace of both sides,

$$\begin{aligned} \mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{2}{s^2} - \frac{1}{s - 2}\right) \\ y(t) &= 1 + 2t - e^{2t} \end{aligned}$$

Note: if we were to have solved this problem using the  $y_c$  and  $y_p$  approach, we would have a case where  $y_c = c_1$  (constant solution), so our assumed form for  $y_p$  would have needed a  $t$  multiplier to avoid the overlap. Using Laplace transforms avoids the need for this special-case logic when you are building the solution.

13.  $x' - 3x = 39 \sin(2t)$ ,  $x(0) = 2$

Taking Laplace of both sides,

$$\underbrace{[sX(s) - 2]}_{\mathcal{L}(x')} - 3X(s) = \frac{39}{s^2 + 4}$$

Group  $X(s)$  terms on left:  $(s - 3)X(s) = 2 + \frac{39}{s^2 + 4}$

$$X(s) = \frac{2}{s - 3} + \frac{39}{(s - 3)(s^2 + 4)}$$

Looking just at the more complicated right-hand term,  $\frac{39}{(s - 3)(s^2 + 4)} = \frac{A}{s - 3} + \frac{Bs + C}{s^2 + 4}$

Solving for  $A$ ,  $B$  and  $C$  gives

$$= \frac{3}{s - 3} + \frac{-3s - 9}{s^2 + 4}$$

Combining with the other term gives

$$\begin{aligned} X(s) &= \left( \frac{2}{s-3} \right) + \left( \frac{3}{s-3} + \frac{-3s-9}{s^2+4} \right) \\ &= \frac{5}{s-3} - 3 \frac{s}{s^2+4} - \frac{9}{2} \frac{2}{s^2+4} \\ \text{so } x(t) &= 5e^{3t} - 3 \cos(2t) - \frac{9}{2} \sin(2t) \end{aligned}$$

14.  $x'' + 6x' + 34x = -34$ ,  $x(0) = 2$ ,  $x'(0) = 3$

Taking Laplace of both sides,

$$\underbrace{[s^2 X(s) - s \cdot 2 - (3)]}_{\mathcal{L}(x'')} + 6 \underbrace{[sX(s) - 0]}_{\mathcal{L}(x')} + 34X(s) = \frac{-34}{s}$$

Group  $X(s)$  terms on left:  $(s^2 + 6s + 34)X(s) = 2s + 3 - \frac{34}{s}$

$$X(s) = \frac{2s + 3}{s^2 + 6s + 34} - \frac{34}{s(s^2 + 6s + 34)}$$

Focusing on the complicated right-hand term,

$$\frac{-34}{s(s^2 + 6s + 34)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 6s + 34}$$

Solving for  $A$  and  $B$  gives

$$= \frac{-1}{s} + \frac{s + 6}{s^2 + 6s + 34}$$

so, taking inverse Laplace of both sides,

$$\begin{aligned} \mathcal{L}^{-1}(X(s)) &= \mathcal{L}^{-1} \left( \frac{2}{s+2} - \frac{2}{s+1} \right) \\ x(t) &= 2e^{-2t} - 2e^{-t} \end{aligned}$$