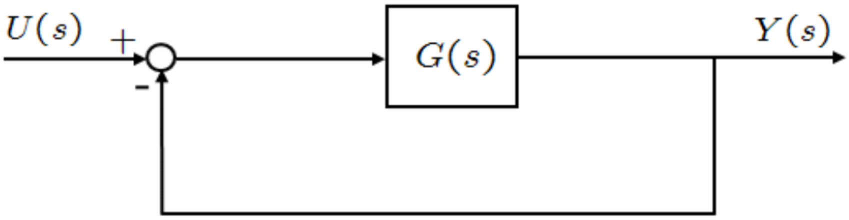
Your Name:	Your Signature:

- **Exam duration:** 2 hours and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place $\boxed{\text{a box around your final answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	30	
2	30	
3	25	
4	15	
5	25	
6	20	
7	55	
Total	200	



1. (30 total points) For the system shown in the above figure, assume that

$$G(s) = K\frac{s^2+4s+8}{s^2-2s}.$$

   (a) (30 points) Sketch the root locus for the above system. You should follow the procedure outlined in the attached formula sheet and **clearly show the individual steps, i.e., Steps 1–10.** If some of the steps are not applicable, state that and explain why they're not.

   Also, you're given that $\arctan(1) = 45\,\text{deg}, \arctan(0.5) = 26.5\,\text{deg}$. You should use these values to compute the angle of arrivals. You are also required to find/approximate the $j\omega$ axis crossing.

1. Poles: $p_{1,2} = 0, +2$. Zeros: $z_{1,2} = -2 \pm 2j$. $n_p = n_z = 2$
2. Asymptotes: none as $n_p - n_z \leq 0$
3. Breakaway points:

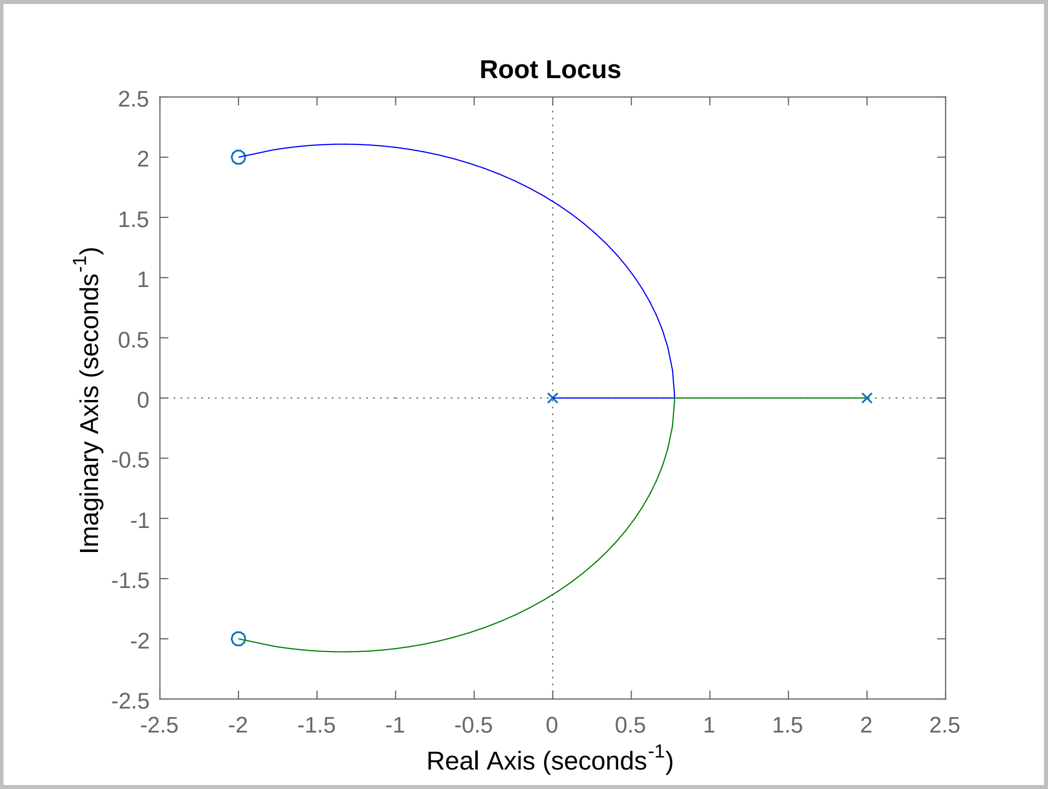
$$\frac{dK}{ds} = 0 \Rightarrow 3s^2 + 8s - 8 = 0 \Rightarrow s_{1,2} = -3.44, 0.775 \Rightarrow s_2 = 0.775 \text{ is a breakaway point}$$

4. Angles of arrival at the complex zeros:

$$\phi_{z_1} = 180 - 90 + (180 - \arctan(2/2)) + (180 - \arctan(2/4)) = 18.43\,\text{deg}$$

$$\Rightarrow \phi_{z_2} = -18.43\,\text{deg}$$

5. $j\omega$ axis crossing: $K \approx 0.475, \omega \approx \pm 1.6$
6. Plot:



2. (30 total points) For a unity feedback system with a compensator, assume that

$$G(s) = \frac{2}{s(0.2s+1)}.$$

   (a) (30 points) Design a lead compensator $G_c(s) = K\frac{s+z}{s+p}$, such that the desired CLTF poles are $s_d = -4 \pm 4j$. You should follow the procedure outlined in the attached formula sheet, and **clearly** show the individual steps, i.e., Steps 1–7.

   Important remark: You are given that $\theta = \angle G(s_d) \approx -210\,\text{deg}$. Show the geometric procedure on the given graph paper.

1. Angle of deficiency: $\phi = -180 - (-210) = 30\,\text{deg}$
2. Following the procedure, we obtain: $z \approx 4.5$, $p \approx 7$
3. Then, we obtain $K \approx 2.9$:
4. Hence:

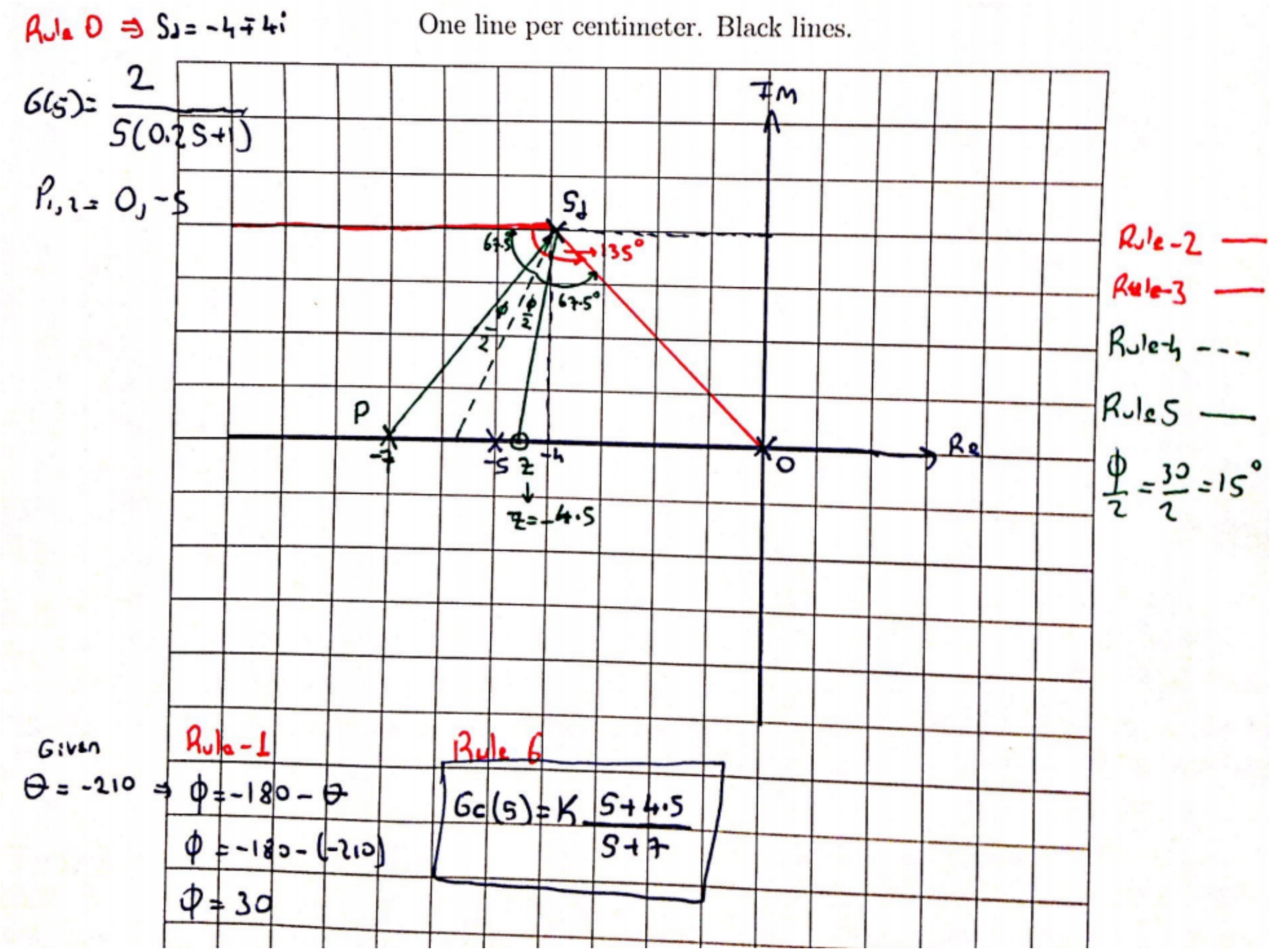
$$G_c(s) = 2.9\frac{s+4.5}{s+7}$$

5. MATLAB verification:

```
clc
clear all
s=tf(’s’)
%s=-4+4*i;
num=[0 0 2];
den=[0.2 1 0];
G=tf(num,den);
H=1;
K=2.9;

num1=[1 4.5];
den1=[1 7];
V=tf(num1,den1);
C=K*V;
TF=feedback(C*G,H)
step(TF)

TF =
5.8 s + 26.1
---------------------------------
0.2 s^3 + 2.4 s^2 + 12.8 s + 26.1
Continuous-time transfer function.

Trial>> p=[ 0.2  2.4   12.8  26.1]
p =
0.2000    2.4000    12.8000    26.1000

Trial>> roots(p)
```

```
ans =
-3.9220 + 4.0023i % Desired Poles (Sd)
-3.9220 - 4.0023i % Desired Poles (Sd)
-4.1560 + 0.0000i
```

6. Plot:



3. (25 total points) Consider the following LTI system:

$$\dot{\boldsymbol{x}}(t) = \underbrace{\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\boldsymbol{C}} x(t).$$

(a) (5 points) Determine the stability of the above system.

Eigenvalues of $\boldsymbol{A}$ are: $\lambda_{1,2} = 2, 3$. Hence, the system is NOT stable.

(b) (5 points) Compute the controllability and observability matrices $\mathcal{C}$ and $\mathcal{O}$. The formula is given in the formula sheet.

$$\mathcal{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \mathcal{O} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}.$$

(c) (5 points) Is the system controllable? Observable? Justify your answer.

The rank of the controllability matrix is equal to $1 < n = 2$, hence the system is not controllable. The observability matrix is full rank, hence the system is observable.

(d) (10 points) From the state-space matrices, obtain the transfer function. The formula is given in the formula sheet.

The transfer function is given by:

$$H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 = \frac{s-3}{s^2 - 5s + 6} = \frac{1}{s-2}.$$

4. (15 total points) A unity feedback system with a plant transfer function:

$$G(s) = \frac{10e^{-2s}}{6s+1}$$

is given.

(a) (15 points) Determine the parameters of the PID controller using one of the Ziegler-Nichols tuning formulas. Obtain the PID controller transfer function in its standard form.

Clearly, the CLTF has an S-shaped step-response which is approximated by a first order delayed TF. The parameters we need here are $K = 10, L = 2, T = 6$. Hence, we can use Ziegler Nichols tuning method 1 to find the parameters for the PID controller:

$$K_p = 1.2\left(\frac{6}{20}\right) = 0.36, T_i = 2L = 4, T_D = 0.5L = 1.$$

Thus, the transfer function for the PID controller can be written as:

$$G_{PID}(s) = 0.36(1 + \frac{1}{4s} + s).$$

5. (25 total points) A unity feedback system with a plant transfer function:

$$G(s) = \frac{2}{s^3 + 7s^2 + 10s}$$

is given.

(a) (25 points) Determine the parameters of the PID controller using the second method of Ziegler-Nichols. You should analytically obtain K_{cr} and P_{cr}. Note that the K_{cr} is computed via the Routh array method, and $P_{cr} = \frac{2\pi}{\omega_{cr}}$. Obtain the PID controller transfer function in its standard form.

1. First, we have to find K_{cr}—the critical gain. Given that a proportional controller is added to the system, the CLTF becomes:

$$\frac{Y(s)}{U(s)} = \frac{KG(s)}{1 + KG(s)} = \frac{2K}{s^3 + 7s^2 + 10s + 2K}.$$

2. Hence, the characteristic polynomial is $s^3 + 7s^2 + 10s + 2K = 0$. To obtain the critical gain K_{cr}, we apply the Routh array method to find the maximum value for K. Use this criterion, we obtain $K_{cr} = 35$.
3. From this K_{cr}, we can compute ω_{cr}. Setting $s = j\omega_{cr}$ in the CP, we obtain:

$$(j\omega_{cr})^3 + 7(j\omega_{cr})^2 + 10(j\omega_{cr}) + 70 = 0 \Rightarrow (70 - 7\omega_{cr}^2) + j(-\omega_{cr}^3 + 10\omega_{cr}) = 0$$

$$\Rightarrow \omega_{cr} = \sqrt{10} \approx 3.16 \Rightarrow P_{cr} = \frac{2\pi}{\omega_{cr}} \approx 2.$$

4. Given the obtained P_{cr} and K_{cr}, we can find the parameters for the PID controller. The PID transfer function can be written as:

$$G_{PID}(s) = 21\left(1 + \frac{1}{s} + 0.25s\right) = 5.25\frac{(s+2)^2}{s}.$$

6. (20 total points) The closed loop transfer function of a system is given as follows:

$$\frac{Y(s)}{U(s)} = \frac{s^2+5s+10}{(s+1)(s+2)(s+3)} = \frac{s^2+5s+10}{s^3+6s^2+11s+6}.$$

is given.

(a) (20 points) Obtain the controllable, observable, and diagonal canonical forms. For the diagonal forms, you have to obtain the residues for the three given poles. Show all the involved steps.

- Clearly, $n = 3, b_0 = 0, b_1 = 1, b_2 = 5, b_3 = 10, a_1 = 6, a_2 = 11$, and $a_3 = 6$.
- Hence, the canonical forms can be written as:
  - *Controllable canonical form:*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}0\\0\\1\end{bmatrix}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \begin{bmatrix}10 & 5 & 1\end{bmatrix}\boldsymbol{x}(t) + 0\boldsymbol{u}(t)\end{aligned}$$

  - *Observable canonical form:*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \begin{bmatrix}0 & 0 & -6\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}10\\5\\1\end{bmatrix}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \begin{bmatrix}0 & 0 & 1\end{bmatrix}\boldsymbol{x}(t) + 0\boldsymbol{u}(t)\end{aligned}$$

  - *Diagonal canonical form:*

$$\frac{Y(s)}{U(s)} = \frac{s^2+5s+10}{(s+1)(s+2)(s+3)} = \frac{3}{s+1} - \frac{4}{s+2} + \frac{2}{s+3}$$

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}1\\1\\1\end{bmatrix}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \begin{bmatrix}3 & -4 & 2\end{bmatrix}\boldsymbol{x}(t) + 0\boldsymbol{u}(t)\end{aligned}$$

7. (55 total points) The state-space representation of a dynamical system is given as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

with

$$\boldsymbol{A} = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}, \boldsymbol{B} = \begin{bmatrix}1\\ -1\end{bmatrix}, \boldsymbol{C} = \begin{bmatrix}2 & 1\end{bmatrix}, \boldsymbol{x}_0 = \begin{bmatrix}-2\\ 3\end{bmatrix}, D = 0.$$

(a) (10 points) By finding the eigenvalues, eigenvectors of the $\boldsymbol{A}$ matrix, compute $e^{\boldsymbol{A}t}$ via the diagonal transformation. You have to clearly show your work.

$$\boldsymbol{A} = \begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}^{-1}$$

$$\Rightarrow e^{\boldsymbol{A}t} = \begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0.5-0.5e^{-2t}\\ 0 & e^{-2t}\end{bmatrix}.$$

(b) (10 points) Assume that the control input is $u(t) = 0$, compute $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$. The initial conditions and state-space matrices are given in the problem description.

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 = \begin{bmatrix}1 & 0.5-0.5e^{-2t}\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}-2\\ 3\end{bmatrix} = \begin{bmatrix}-1.5e^{-2t}-0.5\\ 3e^{-2t}\end{bmatrix}.$$

$$y(t) = \boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}-1.5e^{-2t}-0.5\\ 3e^{-2t}\end{bmatrix} = -1.$$

(c) (25 points) Assume that the control input is $u(t) = 1 + 2e^{-2t}$, compute $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$. The initial conditions and state-space matrices are given in the problem description.

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau.$$

$$\begin{aligned}\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau &= \int_{t_0}^{t} \begin{bmatrix} 1 & 0.5 - 0.5e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1 + 2e^{-2\tau})\,d\tau \\ &= \begin{bmatrix} 0.75 + 0.5t - 0.75e^{-2t} + te^{-2t} \\ -0.5 + 0.5e^{-2t} - 2te^{-2t} \end{bmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}\boldsymbol{x}(t) &= \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.75 + 0.5t - 0.75e^{-2t} + te^{-2t} \\ -0.5 + 0.5e^{-2t} - 2te^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} 0.25 + 0.5t - 2.25e^{-2t} + te^{-2t} \\ -0.5 + 3.5e^{-2t} - 2te^{-2t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\end{aligned}$$

and

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \boldsymbol{x}(t) = t - e^{-2t}.$$

(d) (10 points) Given your answers to the previous question, compute $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ as $t \to \infty$. Which state blows up? Also, find $y(\infty)$.

$$\boldsymbol{x}(\infty) = \begin{bmatrix} \infty \\ -0.5 \end{bmatrix} = \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix}, y(\infty) = \infty.$$

The first state blows up (this state corresponds to the unstable mode with eigenvalue $\lambda_1 = 0$) and the second state converges to -0.5 (this state corresponds to the stable mode with eigenvalue $\lambda_2 = -2$.)