

Your Name:

Your Signature:

- **Exam duration:** 2 hours and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct**.
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- You can ask as many questions as you want.
- This exam has 15 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	15	
2	15	
3	50	
4	15	
5	25	
6	15	
7	45	
8	25	
9	25	
10	20	
Total	250	

1. (15 total points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system.

For each of the following statements, determine if it is true or false. If it is true, **prove** why; if it is false, find a counter example.

- (a) (3 points) The system $\dot{x}(t) = -Ax(t)$ is asymptotically stable.

Solutions: False. Eigenvalues of $-A$ are $-\lambda_i, \forall i = 1, \dots, n$, which are all in the RHP.

- (b) (3 points) The system $\dot{x}(t) = A^\top x(t)$ is asymptotically stable.

Solutions: True. Eigenvalues of A^\top are the same as A .

- (c) (3 points) The system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable (assume A^{-1} exists).

Solutions: True. Eigenvalues of A^{-1} are the same as $\frac{1}{\lambda_i}$ and they're all in the open LHP.

- (d) (3 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.

Solutions: False. Counter example: $A = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}$.

- (e) (3 points) The system $\dot{x}(t) = A^2x(t)$ is asymptotically stable.

Solutions: False. Counter example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. (15 total points) Assume that $x(k+1) = Ax(k)$ is an asymptotically stable discrete-time LTI system.

For each of the following statements, determine if it is true or false. If it is true, explain why; if it is false, find a counter example.

- (a) (3 points) The system $x(k+1) = -Ax(k)$ is asymptotically stable.

Solutions: True. Eigenvalues remain in the unit disk.

- (b) (3 points) The system $x(k+1) = A^T x(k)$ is asymptotically stable.

Solutions: True. Eigenvalues do not change.

- (c) (3 points) The system $x(k+1) = A^{-1}x(k)$ is asymptotically stable (assume A^{-1} exists).

Solutions: False. Eigenvalues becomes larger than 1.

- (d) (3 points) The system $x(k+1) = (A + A^T)x(k)$ is asymptotically stable.

Solutions: False. Counter example: $A = 0.9$.

- (e) (3 points) The system $x(k+1) = A^2x(k)$ is asymptotically stable.

Solutions: True. If $-1 < \lambda_i < 1$, $\Rightarrow 0 < \lambda_i^2 < 1$.

3. (50 total points) Consider three cars moving on the same lane, whose initial locations at time $t = 0$ are $x_1(0) = x_2(0) = x_3(0) = 0$. The above figure exemplifies the movement of cars in 1-D. Suppose the goal is for all three cars to meet at the same location (it does not matter where this meet-up location is). To achieve this goal, the following system dynamics can be designed, where $u(t)$ is an input control for the leading car:

$$\dot{x}_1(t) = x_2(t) - x_1(t) + u(t) \quad (1)$$

$$\dot{x}_2(t) = \frac{x_1(t) + x_3(t)}{2} - x_2(t) \quad (2)$$

$$\dot{x}_3(t) = x_2(t) - x_3(t) \quad (3)$$

In other words, the leading and trailing cars will both move toward the middle car instantaneously; while the middle car will move towards the center of the leading and the trailing cars.

- (a) (5 points) Represent the above dynamics as a CT-LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A, B are matrices that you should determine.

Solutions: Clearly, $A = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- (b) (5 points) Find e^{At} for all $t \in \mathbb{R}$.

Hint: $\begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & -2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

Solutions: $e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & e^{-2t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

- (c) (20 points) Suppose $u(t) = 1, t \geq 0$. Find the expression of $x(t)$ for $t \geq 0$. Also, find $x(t)$ if $t \rightarrow \infty$.

Solution: Given that $u(t) = 1 \forall t \geq 0$, we obtain:

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & & \\ & e^{-2(t-\tau)} & \\ & & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5(1 - e^{-2t}) & \\ & & 1 - e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \end{aligned}$$

- (d) (3 points) Describe the steady-state behaviors of $x_i(t), i = 1, 2, 3$. Your description must have physical, applied meaning.

Solutions: If $t \rightarrow \infty$, we obtain:

$$x(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.25t + 0.625 \\ 0.25t - 0.125 \\ 0.25t - 0.375 \end{bmatrix}$$

In the steady state, all three cars move at the constant velocity 0.25 to the right, with car 1 still in the lead and car 3 still trailing, and the distance between car 1 and car 2 is 0.75 while the distance between car 2 and car 3 is 0.25.

- (e) (2 points) What is the constant velocity for the three cars?

Solutions: 0.25.

- (f) (15 points) Assess the following CT-LTI system properties: stability, controllability, stabilizability, observability, and detectability, given that the outputs for the system are the first two states, i.e., $x_1(t)$ and $x_2(t)$. You'll have to obtain the C-matrix.

Solutions (3pts each):

- Stability:* system is **not** asymptotically stable, as one eigenvalue is equal to 0—on the $j\omega$ -axis.
- Controllability:* the controllability matrix of the given LTI system is $\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ and it's full rank, hence the system is fully controllable.
- Stabilizability:* an LTI system that is fully controllable is stabilizable.
- Observability:* the observability matrix of the given LTI system is $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ and it's full rank, hence the system is fully observable.
- Detectability:* an LTI system that is fully observable is by definition detectable.

4. (15 total points) Suppose in the above graph, one starts from node A on the left and tries to reach node B on the right by only moving to the right at each step. The cost of any path is the sum of the following:
- the cost of all the edges it passes through as indicated by the numbers above the edges
 - the cost of all the intermediate nodes it visits as indicated by the numbers inside the circles.
- (a) (15 points) Use the dynamic programming method to find the path from A to B with the smallest cost. You should only use dynamic programming to solve the above problem. You receive **no credit** if you don't show how you applied DP. Once you're done, draw a diagram/map that shows the DP-solution that exemplifies your numerical solution.

Solutions:

5. (25 total points) Consider the following CT-LTI model of a dynamical system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

with the following cost function:

$$J = \int_0^{\infty} \left(x^{\top}(t)x(t) + u^2(t) \right) dt.$$

(a) (20 points) Find the linear state-feedback control law that minimizes J .

Solutions: To find the optimal controller for this problem, we first solve the algebraic Riccati (ARE) equation we discussed in Module 05. This equation defines the infinite horizon, steady-state solution for CARE—as the final time, t_f , is equal to infinity.

The problem given is as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 1.$$

The optimal linear state-feedback controller is given by:

$$u^*(t) = -Kx(t) = -R^{-1}B^{\top}P^*x(t),$$

where P^* is a 2by2 positive definite matrix solution to ARE:

$$A^{\top}P + PA + Q - PBR^{-1}B^{\top} = 0.$$

Letting $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, we obtain:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving the above equation, we obtain:

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Other solutions for P are invalid as P has to be a positive definite matrix. Therefore, the optimal state-feedback control law is:

$$u^*(t) = -Kx(t) = -R^{-1}B^{\top}P^*x(t) = -x_1(t) - 3x_2(t).$$

(b) (5 points) Find the value of the performance index for the closed-loop system.

Solutions: In homework 5, we derived the value for the performance index for the optimal control problem driven by the state-feedback controller. The performance index value is:

$$J = x_0^\top P x_0 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 15.$$

You are almost halfway through the exam. It might be a good idea to rehydrate and walk for a minute or two...

...Unless you're superman/superwoman...

...You're not superman/superwoman.

6. (15 total points) Compute $x = x(t)$ that solves the following optimal control problem:

$$\text{minimize } J(u) = \int_1^2 \sqrt{1 + u^2(t)} dt$$

$$\text{subject to } \dot{x}(t) = u(t), \quad x(1) = 0, \quad x(2) = 5.$$

(a) (15 points) Using the HJB-equation, find the linear state-feedback control law that minimizes J .

Solutions: First, we construct the Hamiltonian:

$$\mathcal{H}(x, u, J_x, t) = \sqrt{1 + u^2(t)} + \lambda(x, t)u(t).$$

Since there are no constraints on $u(t)$, the optimal controller candidate is:

$$0 = \frac{\partial \mathcal{H}}{\partial u} = \lambda(x, t) + \frac{u}{\sqrt{1 + u^2}} \Rightarrow u^*(t) = \frac{\lambda(x, t)}{\sqrt{1 - \lambda^2(x, t)}}.$$

Now, we formulate the HJB equation:

$$-V_t(x, t) = \left(\frac{\partial \mathcal{H}}{\partial x} \right)^\top = 0 \Rightarrow V(t, x) = v \text{ is constant.}$$

Therefore, $u^*(t) = \frac{\lambda(x, t)}{\sqrt{1 - \lambda^2(x, t)}} = \frac{\lambda}{\sqrt{1 - \lambda^2}} = c$ is also constant. Since $x(1)$ and $x(2)$ are given, we can determine $u^*(t) = c$, as follows:

$$x(2) = x(1) + \int_1^2 c d\tau, \Rightarrow u^*(t) = c = 5.$$

Therefore,

$$x(t) = \int_1^t 5 d\tau = 5t - 5.$$

7. (45 total points) For the following dynamical system under unknown inputs,

$$\begin{aligned}\dot{x}_p(t) &= A_p(t)x_p(t) + B_p^{(1)}u_1(t) + B_p^{(2)}u_2(t) \\ y(t) &= C_px_p(t),\end{aligned}$$

a sliding-mode observer (SMO) can be designed with the following dynamics:

$$\begin{aligned}\dot{\hat{x}}_p(t) &= A_p\hat{x}_p(t) + B_p^{(1)}u_1(t) + L(y(t) - \hat{y}(t)) - B_p^{(2)}E(\hat{y}, y, \eta) \\ \hat{y}(t) &= C_p\hat{x}_p(t),\end{aligned}$$

where $E(\cdot)$ is defined as (η is SMO gain):

$$E(\hat{y}, y, \eta) = \begin{cases} \eta \frac{F(\hat{y} - y)}{\|F(\hat{y} - y)\|_2}, & \text{if } F(\hat{y} - y) \neq 0 \\ 0, & \text{if } F(\hat{y} - y) = 0. \end{cases}$$

(a) (15 points) The SMO design objective is to find matrices $P = P^\top \succ 0, F$ and L that satisfy the following equations for a predefined $Q = Q^\top \succ 0$:

$$FC_p = (B_p^{(2)})^\top P$$

$$(A_p - LC_p)^\top P + P(A_p - LC_p) = -Q$$

Are the above two equations linear matrix inequalities? If no, formulate the above equations as a set of linear matrix inequalities.

Solutions: The We have two (nonlinear) matrix equations in terms of P, F, L :

$$(A_p - LC_p)^\top P + P(A_p - LC_p) = -Q$$

$$P = P^\top$$

$$FC_p = (B_p^{(2)})^\top P$$

We use a simple LMI trick. Set $Y = PL$, rewrite above system of **linear** matrix equations as:

$$A_p^\top P + PA_p - C_p^\top Y^\top - YC_p = -Q$$

$$P = P^\top$$

$$FC_p = (B_p^{(2)})^\top P$$

From the solutions of P, Y , we can solve for L : $L = P^{-1}Y$. Hence, the SMO design problem can be written as an LMI, albeit we have more variables to solve for.

(b) (15 points) Write a CVX script to solve the SMO design problem, written as an LMI.

Solutions: Sample CVX code—

```

cvx_clear
cvx_begin sdp quiet

variable P(n, n) symmetric
variable Y(n, p)
variable F(m2, p)

minimize(0)

subject to

Ap'*P + P*Ap - Y*Cp - Cp'*Y' <= 0
F*Cp-Bp2'*P==0;
P >= 0

cvx_end

L = P\Y;

```

- (c) (15 points) If your state-estimates are converging, how can you reconstruct (or estimate) the vector of unknown inputs from the system dynamics? You only need this equation. Think about it—the solution is very simple.

$$\dot{x}_p(t) = A_p x_p(t) + B_p^{(1)} u_1(t) + B_p^{(2)} u_2(t)$$

Hint: You should first discretize the above dynamical system and then use $\hat{x}(t)$ to obtain $u_2(t)$.

Solutions: Since the state-estimates are converging, then $\hat{x}(t) \rightarrow x(t)$ or $\hat{x}(k) \rightarrow x(k)$. Leveraging the given hint, we can obtain the discretized version of the system dynamics, as follows:

$$x(k+1) \approx \hat{x}(k+1) = A_d \hat{x}(k) + B_p d^{(1d)} u_1(k) + B_p d^{(2)} \hat{u}_2(k).$$

Since the estimation is accurate, and since we have $u_1(k)$, $A_d, B_p d$ we can obtain an estimate of the unknown input vector $u_2(k)$ as follows:

$$\hat{u}_2(k) = \left(B_p d^{(2)} \right)^\dagger \left(\hat{x}(k+1) - A_d \hat{x}(k) - B_p d^{(1d)} u_1(k) \right),$$

where $\left(B_p d^{(2)} \right)^\dagger$ is the pseudo-inverse of $B_p d^{(2)}$, which means we need $B_p d^{(2)}$ to be full-column rank.

8. (25 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\dot{x}_p = A_p x_p + B_p \hat{u} \quad (4)$$

$$y = C_p x_p \quad (5)$$

$$\dot{x}_c = A_c x_c + B_c \hat{y} \quad (6)$$

$$u = C_c x_c + D_c \hat{y}, \quad (7)$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

Figure 1: NCS architecture.

The network-induced error state is defined as follows:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} = \begin{bmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{bmatrix}.$$

(a) (25 points) Derive the dynamics of the combined NCS state:

$$z(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \\ e_y(t) \\ e_u(t) \end{bmatrix},$$

i.e., derive

$$\dot{z}(t) = Az(t)$$

where A is a matrix you should determine in terms of $A_p, B_p, C_p, A_c, B_c, C_c, D_c$ only, given that ZOH is considered for signals \hat{y} and \hat{u} .

Solutions: Check Module 08. The overall state of the NCS is:

$$x(t)^\top = [x_p(t)^\top \ x_c(t)^\top]$$

Without a network, $e(t) = 0$, dynamics are reduced to:

$$\dot{x}(t) = A_{11}x(t), \quad A_{11} = \begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_c \end{bmatrix}.$$

With the network, $e(t) \neq 0$, define network-induced error state:

$$e(t)^\top = [\hat{y}(t)^\top \ \hat{u}(t)^\top] - [y(t)^\top \ u(t)^\top].$$

Hence, the combined NCS state can be written as:

$$z(t)^\top = [x(t)^\top \ e(t)^\top].$$

Given the plant and controller dynamics, the NCS overall state dynamics are obtained as follows:

$$\dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = Az(t).$$

$$A_{12} = \begin{bmatrix} B_p D_c & B_p \\ B_c & 0 \end{bmatrix}, A_{21} = - \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix} A_{11}, A_{22} = - \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix} A_{12}.$$

9. (25) The following optimization problem is given:

$$\underset{x}{\text{minimize}} \frac{x^\top Qx}{x^\top Px},$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hint: set the denominator to 1, and make this a constraint to the optimization problem. Then, you'll get an equality-constrained optimization problem, and hence you can use the KKT conditions we discussed in class.

(a) (25 points) Solve the above optimization problem.

Solutions: Given the hint, the problem becomes:

$$\begin{aligned} &\text{minimize} && x^\top Qx \\ &\text{subject to} && 1 - x^\top Px = 0. \end{aligned}$$

We now formulate the Lagrangian of the optimization problem:

$$\mathcal{L}(x, \lambda) = x^\top Qx + \lambda(1 - x^\top Px).$$

The first-order necessary conditions of optimality can be written produces:

$$\nabla_x \mathcal{L}(x, \lambda) = 2Qx - 2\lambda Px = 0,$$

or

$$(\lambda I_2 - P^{-1}Q)x = 0 \rightarrow \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) x \rightarrow \begin{bmatrix} \lambda - 1 & \\ & \lambda - 2 \end{bmatrix} x = 0.$$

Therefore, $\lambda_1 = 1, \lambda_2 = 2$ and the corresponding solutions are:

$$x^{(1)} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x^{(2)} = \pm \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \end{bmatrix}.$$

By simply comparing λ_1 and λ_2 and their corresponding solutions, we find that $\lambda_1 = 1$ is optimal in comparison with $\lambda_2 = 2$. Checking the second-order conditions, we obtain the Hessian of $\mathcal{L}(x, \lambda)$:

$$\nabla_x^2 \mathcal{L}(x^{(1)}, \lambda_1) = \begin{bmatrix} 0 & \\ & 4 \end{bmatrix},$$

which is positive semi-definite. Finding the tangent space and checking the positive-definiteness of the Hessian on the tangent space, we conclude that $x^{(1)} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a strict minimizer for the given optimization problem.

10. (20) The following optimization problem is given:

$$\begin{aligned} & \text{maximize} && x_1^2 + 4x_2^2 \\ & \text{subject to} && x_1^2 + 2x_2^2 \leq 2 \end{aligned}$$

(a) (10 points) Derive the KKT conditions and find the set of points satisfying these conditions.

Solutions: The Lagrangian can be written as:

$$\mathcal{L}(x, \mu) = -(x_1^2 + 4x_2^2) + \mu(x_1^2 + 2x_2^2 - 2).$$

Applying the KKT conditions, we obtain the following candidates:

- $x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mu^{(1)} = 0$
- $x^{(2)} = \pm \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \mu^{(2)} = 1$
- $x^{(3)} = \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mu^{(3)} = 2$

(b) (10 points) Apply the second order necessary conditions to determine whether the KKT conditions-satisfying points are minimizers or not.

Solutions: Applying the SONC, we obtain the following:

- $x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a strict local minimizer of the problem;
- $x^{(2)} = \pm \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ doesn't satisfy the SONC;
- $x^{(3)} = \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a strict local maximizer of the problem.