Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

| Question Number | Maximum Points | Your Score |
|---|---|---|
| 1 | 45 | |
| 2 | 35 | |
| 3 | 20 | |
| ***Total*** | 100 | |

1. (45 total points) Answer the following unrelated miscellaneous questions.

   (a) (10 points) Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t)\\ \dot{x}_2(t) &= x_1(t) - x_2(t) - 1.\end{aligned}$$

Find **two** equilibrium points of the nonlinear system. By two equilibirum points I mean:

$$\boldsymbol{x}_e^{(1)} = \begin{bmatrix} x_{e1}^{(1)} \\ x_{e2}^{(1)} \end{bmatrix}, \textbf{ and } \boldsymbol{x}_e^{(2)} = \begin{bmatrix} x_{e1}^{(2)} \\ x_{e2}^{(2)} \end{bmatrix}.$$

The equilibrium points for this system are:

- $x_e^{(1)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, x_e^{(2)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

   (b) (10 points) You are given a matrix $\boldsymbol{A}$ with the characteristic polynomial

$$\pi_{\boldsymbol{A}}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)^4 = 0.$$

In other words, $\boldsymbol{A}$ has three distinct eigenvalues $\lambda_{1,2,3}$ of different algebraic multiplicity. Given that

$$\dim \mathcal{N}(\boldsymbol{A} - \lambda_2 \boldsymbol{I}) = 2, \quad \dim \mathcal{N}(\boldsymbol{A} - \lambda_3 \boldsymbol{I}) = 2,$$

obtain **all possible Jordan canonical forms** for $\boldsymbol{A}$. You have to be clear and precise. Explain your answer.

The dimension of the nullspace for each eigenvector determines the number of Jordan blocks for eigenvalues $\lambda_2$ and $\lambda_3$:

- For eigenvalue $\lambda_1$, the only possible Jordan block is

$$J_{\lambda_1} = \begin{bmatrix} \lambda_1 \end{bmatrix}.$$

- For eigenvalue $\lambda_2$, the only possible Jordan block is $\boldsymbol{J}_{\lambda_2} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ since the geometric multiplicity is equal to the algebraic one, then there will be two Jordan blocks for $\lambda_2$. Since the total size of these two Jordan blocks is equal to 2, then the only possible Jordan block form for $\lambda_2$ is

$$\boldsymbol{J}_{\lambda_2} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

- For eigenvalue $\lambda_3$, the geometric multiplicity is equal to 2, hence there are two Jordan blocks with a total size of 4. The possible combinations are hence

$$\boldsymbol{J}_{\lambda_2}^{(1)} = \begin{bmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix},$$

or

$$J^{(2)}_{\lambda_2} = \begin{bmatrix} \lambda_3 & 1 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}.$$

Therefore, and given the problem description, there can only be two possible combinations of the Jordan form of $\boldsymbol{A}$, given as follows:

$$\boldsymbol{J}^{(1)} = \text{blkdiag}(J_{\lambda_1}, \boldsymbol{J}_{\lambda_2}, \boldsymbol{J}^{(1)}_{\lambda_3})$$

or

$$\boldsymbol{J}^{(2)} = \text{blkdiag}(J_{\lambda_1}, \boldsymbol{J}_{\lambda_2}, \boldsymbol{J}^{(2)}_{\lambda_3}).$$

(c) (10 points) Consider that

$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^\top = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}.$$

Note that $\boldsymbol{A}$ is a rank one matrix.

**Derive** $e^{\boldsymbol{A}t}$ for any $\boldsymbol{u}$ and $\boldsymbol{v}$ and then compute $e^{\boldsymbol{A}t}$ for the matrix given above and for $t = \dfrac{1}{\boldsymbol{v}^\top \boldsymbol{u}} = \dfrac{1}{32}$.

If $\boldsymbol{A}$ is a rank-1 matrix, we can write

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \frac{\boldsymbol{A}}{\boldsymbol{v}^\top \boldsymbol{u}} \left[ e^{(\boldsymbol{v}^\top \boldsymbol{u})t} - 1 \right].$$

Notice that

$$\boldsymbol{v}^\top \boldsymbol{u} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32,$$

hence

$$e^{\boldsymbol{A}t} = \boldsymbol{I}_3 + \frac{\boldsymbol{A}}{\boldsymbol{v}^\top \boldsymbol{u}} \left[ e^{(\boldsymbol{v}^\top \boldsymbol{u})t} - 1 \right] = \boldsymbol{I} + \frac{\boldsymbol{A}}{32} \left[ e^1 - 1 \right] \approx \boldsymbol{I} + 0.05\boldsymbol{A}.$$

(d) (10 points) Is the following system defined by

$$y(t) = (u(t))^{1.1} + u(t+1)$$

causal or non-causal? Linear or nonlinear? Time-invariant or time-varying? You have to prove your answers. A one-word answer is not enough.

The system is nonlinear due to the $(u(t))^{1.1}$ (which is a nonlinear function in terms of the input), causal because the output depends on future inputs, and time-invariant. You have to prove these results, though. :)

(e) (5 points) The transfer function matrix of the state space system

$$\dot{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

can be written as

$$\boldsymbol{H}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

for any MIMO or SISO system. Find the transfer function $\boldsymbol{H}(s)$ when

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Your $\boldsymbol{H}(s)$ should be $\in \mathbb{R}^{1\times 2}$

$$\boldsymbol{H}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left( \begin{bmatrix} s & 0 \\ 0s & \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{s-2} & 0 \end{bmatrix}.$$

2. (35 total points) The state-space representation of a dynamical system is given as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \boldsymbol{x}_0 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, D = 0.$$

(a) (5 points) By finding the eigenvalues, eigenvectors of the $\boldsymbol{A}$ matrix, compute $e^{\boldsymbol{A}t}$ via the diagonal transformation. You have to clearly show your work.

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1}$$

$$\Rightarrow e^{\boldsymbol{A}t} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0.5 - 0.5e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}.$$

(b) (5 points) Assume that the control input is $u(t) = 0$, compute $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$.

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 = \begin{bmatrix} 1 & 0.5 - 0.5e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix}.$$

$$y(t) = \boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} = -1.$$

(c) (20 points) Assume that the input is $u(t) = 1 + 2e^{-2t}$, compute $\boldsymbol{x}(t)$, $y(t)$.

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau.$$

$$\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau = \int_{t_0}^{t} \begin{bmatrix} 1 & 0.5 - 0.5e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1 + 2e^{-2\tau})\,d\tau$$

$$= \begin{bmatrix} 0.75 + 0.5t - 0.75e^{-2t} + te^{-2t} \\ -0.5 + 0.5e^{-2t} - 2te^{-2t} \end{bmatrix}.$$

Hence,

$$\boldsymbol{x}(t) = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.75 + 0.5t - 0.75e^{-2t} + te^{-2t} \\ -0.5 + 0.5e^{-2t} - 2te^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 + 0.5t - 2.25e^{-2t} + te^{-2t} \\ -0.5 + 3.5e^{-2t} - 2te^{-2t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \boldsymbol{x}(t) = t - e^{-2t}.$$

(d) (5 points) Given your answers to the previous question, compute $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ as $t \to \infty$. Which state blows up? Also, find $y(\infty)$.

$$\boldsymbol{x}(\infty) = \begin{bmatrix} \infty \\ -0.5 \end{bmatrix} = \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix}, y(\infty) = \infty.$$

The first state blows up (this state corresponds to the unstable mode with eigenvalue $\lambda_1 = 0$) and the second state converges to -0.5 (this state corresponds to the stable mode with eigenvalue $\lambda_2 = -2$.)

3. (20 total points) In this problem, we will study the equilibrium of Susceptible-Infectious-Susceptible (SIS) in epidemics—similar to what we discussed in class. The dynamics of a simplified SIS model can be written as

$$\frac{dS}{dt} = -\frac{\beta SI}{N(t)} + \gamma I \quad (1)$$

$$\frac{dI}{dt} = \frac{\beta SI}{N(t)} - \gamma I \quad (2)$$

where $S(t)$ is the number of people that are susceptible at time $t$ and $I(t)$ is the number of infected people at time $t$, where $N(t)$ is the total number of people which is **a time-varying quantity**.

Assume that the number of people is fixed, that is $S(t) + I(t) = N(t)$ where $N(t)$ is the **time-varying population** of the SIS dynamics.

(a) (10 points) Given the above assumption, reduce the above dynamical system from 2 states $(S(t), I(t))$ to a dynamic system with only one state $I(t)$. You should obtain something like

$$\dot{I}(t) = f(I(t), \beta, N(t), \gamma)$$

where $f(\cdot)$ is the function that you should determine.

(b) (5 points) What is the non-trivial (different than zero) **time-varying equilibrium** of the system? In other words, what is $I_{eq}(t)$?

(c) (5 points) The linearized dynamics of $I(t)$ can be written as:

$$\dot{I}_{lin}(t) = \left.\frac{\partial f(t)}{\partial I(t)}\right|_{I(t)=I_{eq}(t)} \cdot I_{lin}(t).$$

where $\Big|_{I(t)=I_{eq}(t)}$ means "evaluated at $I(t) = I_{eq}(t)$". In other words, the linearized dynamic system can be written as

$$\dot{x}(t) = \alpha(t) \cdot x(t)$$

where $x(t)$ is the linearized state $I_{lin}(t)$, and $\alpha(t) = \left.\frac{\partial f(t)}{\partial I(t)}\right|_{I(t)=I_{eq}(t)}$. Analyze the stability of this equilibrium point and explain what happens as $t \rightarrow$ as any of these parameters $\beta, N(t), \gamma$ change.

Does the stability of the linearized systme depend on $N(t)$?