Your Name: ____________________ Your Signature: ____________________

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place [a box around your final answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 6 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	30	
2	25	
3	20	
4	15	
5	10	
Total	100	

1. (30 total points) Answer the following, unrelated miscellaneous questions.

   (a) (10 points) Consider the nonlinear system given by this differential equation:

$$\ddot{y}(t) - (1 - y^2(t))\dot{y}(t) + y(t) - 2 = 0.$$

   - Write the system in a state-space format:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}), \quad y(t) = g(\boldsymbol{x}),$$

     where $f(x)$ and $g(x)$ are the state space functions, and $\boldsymbol{x}(t)$ is the state vector of size 2.
   - Find the equilibrium to the above nonlinear system.

- State space format:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_2(t) \\ (1 - x_1^2(t))x_2(t) - x_1(t) + 2 \end{bmatrix}, \quad y(t) = x_1(t).$$

- $x_1^* = 2, x_2^* = 0$

(b) (10 points) Find the Jordan canonical form and the matrix exponential of

$$\boldsymbol{A} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix},$$

given that $\lambda = -1$ is the only eigenvalue of $\boldsymbol{A}$. You can write $e^{\boldsymbol{A}t}$ as $\boldsymbol{T}e^{\boldsymbol{J}t}\boldsymbol{T}^{-1}$, i.e., no need to multiply the three matrices or to compute $\boldsymbol{T}^{-1}$.

- The matrix has three evalues at -1. Hence, the algebraic multiplicity of $\lambda = -1$ is equal to 3.
- First, we find the evectors corresponding to λ:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0} \Rightarrow \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- Next, we find the generalized evectors for λ. Note that we only have one Jordan block of size 3. The generalized evectors $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are:

$$\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

- The Jordan matrix is:

$$\boldsymbol{J} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

- The factorization then yields:

$$\boldsymbol{A} = \boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^{-1},$$

where $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$.
- The matrix exponential is given by:

$$e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{J}t}\boldsymbol{T}^{-1}, \quad e^{\boldsymbol{J}t} = \begin{bmatrix} e^{-t} & te^{-t} & 0.5t^2e^{-t} \\ 0 & e^{-t} & te^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}.$$

(c) (10 points) Is the system defined by

$$y(t) = N(u(t)) = |u(t)| + \alpha(t)u(t)$$

linear or nonlinear? Time varying or time-invariant? Prove your answers.

- Nonlinear and time-varying.
- The mapping is nonlinear since

$$N(\alpha_1 u_1(t) + \alpha_2 u_2(t)) \neq \alpha_1 N(u_1(t)) + \alpha_2 N(u_2(t))$$

  due to the absolute value function in the mapping.
- The mapping is time-varying due to the $\alpha(t)$ term—as discussed in class.

2. (25 total points) You are given the following LTI system with two inputs:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

where

$$\boldsymbol{A} = \boldsymbol{T}\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}\boldsymbol{T}^{-1}, \quad \boldsymbol{B} = \boldsymbol{T}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix}, \quad a, b \neq 0.$$

(a) (5 points) Find a general expression for $e^{\boldsymbol{A}t}$?

- $e^{\boldsymbol{A}t} = \boldsymbol{T}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{bt} \end{bmatrix}\boldsymbol{T}^{-1}$

(b) (10 points) Obtain a general expression for $\boldsymbol{x}(t)$ starting from any initial condition $(\boldsymbol{x}(t_0))$, given that $\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad t \geq 0$. You should evaluate the integral.

- Final answer:

$$\boldsymbol{x}(t) = \boldsymbol{T}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{a(t-t_0)} & 0 \\ 0 & 0 & e^{b(t-t_0)} \end{bmatrix}\boldsymbol{T}^{-1}\boldsymbol{x}(t_0) + \boldsymbol{T}\begin{bmatrix} 0 \\ 0 \\ be^{bt}(t-t_0) \end{bmatrix}.$$

(c) (10 points) Compute $\boldsymbol{x}(0)$, given that $\boldsymbol{x}(2) = \boldsymbol{T}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

- Final answer:

$$\boldsymbol{x}(t) = \boldsymbol{T}\begin{bmatrix} 1 \\ 2e^{-2a} \\ 3e^{-2b} \end{bmatrix} - 2\boldsymbol{T}\begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}.$$

3. (20 total points) A transfer function of a linear system is given by:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 4s + 0.5}{0.5s^3 + 8s^2 + 16s + 22}.$$

(a) (20 points) Derive the state-space controllable canonical form for this system. You should derive the canonical form and state space matrices, rather than listing them.

- Derivation from Ogata. Note that

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 4s + 0.5}{0.5s^3 + 8s^2 + 16s + 22} = \frac{4s^3 + 0s^2 + 8s + 1}{s^3 + 16s^2 + 32s + 44}.$$

- Final form:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -44 & -32 & -16 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} -175 & -120 & -64 \end{bmatrix}, \quad D = 4.$$

4. (15 total points) In Module 02, we discussed the prey-predator dynamical model of the interaction of two species. This model can be altered as follows:

$$\dot{x}(t) = ax(t) - c\frac{x(t)y(t)}{1 + d \cdot x(t)} \quad (1)$$

$$\dot{y}(t) = -by(t) + c\frac{x(t)y(t)}{1 + d \cdot x(t)}, \quad (2)$$

where $x(t)$ and $y(t)$ denote the population of the two species (prey and predator), and a, b, c, d are all positive constants. This model accounts for the saturation in the population of the prey.

   (a) (15 points) Obtain the nonzero equilibrium solution for the above dynamical model. Your answer should be simplified. The equilibrium quantities x^*, y^* should be given in terms of a, b, c, d.

- Final answer:

$$x^* = \frac{b}{c - bd}, \quad y^* = \frac{a}{c - bd}$$

5. (10 total points) The following systems are given:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \quad (3)$$

$$\dot{\hat{\boldsymbol{x}}}(t) = \hat{\boldsymbol{A}}\hat{\boldsymbol{x}}(t). \quad (4)$$

   (a) (10 points) Assume that $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$ are both square matrices of size n. What is a condition on $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$ that would guarantee that the inner product of their state trajectory would be constant for all $t \geq 0$?

$$\langle \boldsymbol{x}(t), \hat{\boldsymbol{x}}(t) \rangle = \langle \boldsymbol{x}(0), \hat{\boldsymbol{x}}(0) \rangle = \text{constant}.$$

   Note that the inner product of two vectors is: $\langle x, y \rangle = x^\top y$. The condition you derive on $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$ should be only in terms of $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$, not in terms their exponentials.

- Final solution:

$$\hat{\boldsymbol{A}} = -\boldsymbol{A}^\top$$