Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 9 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	25	
2	30	
3	25	
4	20	
Total	100	

1. (25 total points) Answer the following unrelated miscellaneous questions.

   (a) (10 points) Is the system defined by

$$y(t) = N(u(t)) = u(t-3) + \alpha(t)u(t)$$

linear or nonlinear? Time varying or time-invariant? Prove your answers.

- Linear and time-varying.
- The mapping is linear since

$$N(\alpha_1 u_1(t) + \alpha_2 u_2(t)) \neq \alpha_1 N(u_1(t)) + \alpha_2 N(u_2(t)).$$

- The mapping is time-varying due to the $\alpha(t)$ term—as discussed in class.

   (b) (10 points) The below figure shows two interacting tanks in series with outlet flow rate being function of the square root of tank height. The dynamic model is given as:

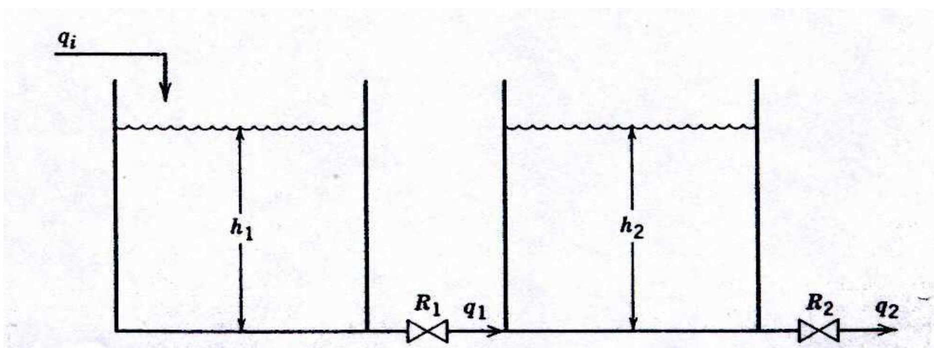

Figure 1: Interaction of two tanks in series.

$$\begin{aligned}\dot{h}_1(t) &= \frac{F(t)}{A_1} - \frac{R_1}{A_1}\sqrt{h_1(t) - h_2(t)} = f_1(h_1, h_2, F)\\ \dot{h}_2(t) &= \frac{R_1}{A_2}\sqrt{h_1(t) - h_2(t)} - \frac{R_2}{A_2}\sqrt{h_2(t)} = f_2(h_1, h_2)\end{aligned}$$

where $h_{1,2}(t)$ are the two states, $F(t)$ is the control input, and A_1, A_2, R_1, and R_2 are all constants. Assume that the equilibrium quantities for states h_1^e, h_2^e, and F^e are all given (i.e., they are fixed). Find the linearized state space representation of the two-tank system.

$$A = \begin{bmatrix} \frac{-R_1}{2A_1\sqrt{h_1^e - h_2^e}} & \frac{R_1}{2A_1\sqrt{h_1^e - h_2^e}} \\ \frac{R_1}{2A_2\sqrt{h_1^e - h_2^e}} & \frac{-R_1}{2A_2\sqrt{h_1^e - h_2^e}} - \frac{R_2}{2A_2\sqrt{h_2^e}} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix}$$

(c) (5 points) Write a pseudo code to generate solutions for the continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

given arbitrary initial conditions, state-space matrices, and control inputs. You can either use the pseudo code for the `ode` solver **or** through the discretization.

- `ode45` code is from the homework
- Discretization code is given as follows:

```
A = ...;
B = ...;
C = ...;
D= ...;
T=sampling_time = ...;
%%
A_tilde=expm(A*t);
n = size(A);
B_tilde=A*(inv(A_tilde)-eye(n))*B;
C_tilde = C;
D_tilde = D;
[p n]=size(C);
x_0 = ... ;
T_final = ...;
T=0:1:T_final;
X=zeros(n,length(T));
Y=zeros(p,length(T));
X(:,0)=x_0;
%%
for k=1:1:length(T)
u(k)=...;%% function of k
X(:,k+1)=A_tilde*X(:,k)+B_tilde*u(k) ;
Y(:,k)=C_tilde*X(:,k)+D_tilde*u(k);
end
```

2. (30 total points) You are given the following LTI system with two inputs:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where

$$A = TDT^{-1} = T\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T^{-1}, \quad B = T\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \\ ab & 0 \end{bmatrix}, \quad a \neq b, a, b \neq 0.$$

(a) (5 points) Find a general expression for e^{At} and $e^{A(t-t_0)}$. Note that A is given in the diagonal form.

- $e^{At} = T\begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{at} & 0 & 0 \\ 0 & 0 & e^{bt} & 0 \\ 0 & 0 & 0 & e^{t} \end{bmatrix} T^{-1}$
- $e^{A(t-t_0)} = T\begin{bmatrix} e^{t_0-t} & 0 & 0 & 0 \\ 0 & e^{a(t-t_0)} & 0 & 0 \\ 0 & 0 & e^{b(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{t-t_0} \end{bmatrix} T^{-1}$

(b) (15 points) Obtain a general expression for $x(t)$ starting from any initial condition $(x(t_0))$, given that $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = e^{bt}\begin{bmatrix} a \\ 1 \end{bmatrix}, \quad t \geq 0$. You should evaluate the integral.

- Final answer:

$$
\begin{aligned}
x(t) &= \overbrace{T\begin{bmatrix} e^{t_0-t} & 0 & 0 & 0 \\ 0 & e^{a(t-t_0)} & 0 & 0 \\ 0 & 0 & e^{b(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{t-t_0} \end{bmatrix} T^{-1}x(t_0)}^{x_{zir}(t)} \\
&\quad + \int_{t_0}^{t} \begin{bmatrix} e^{\tau-t} & & & \\ & e^{a(t-\tau)} & & \\ & & e^{b(t-\tau)} & \\ & & & e^{t-\tau} \end{bmatrix} T^{-1}T \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \\ ab & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} e^{b\tau}\, d\tau \\
&= x_{zir}(t) + T\int_{t_0}^{t} \begin{bmatrix} e^{\tau-t} & & & \\ & e^{a(t-\tau)} & & \\ & & e^{b(t-\tau)} & \\ & & & e^{t-\tau} \end{bmatrix} \begin{bmatrix} a \\ 0 \\ b \\ a^2b \end{bmatrix} e^{b\tau}\, d\tau \\
&= x_{zir}(t) + T\int_{t_0}^{t} \begin{bmatrix} ae^{-t+(b+1)\tau} \\ 0 \\ be^{bt} \\ a^2be^{t+(b-1)\tau} \end{bmatrix} d\tau \\
x(t) &= x_{zir}(t) + T\begin{bmatrix} \alpha\left(e^{bt} - e^{-t+t_0(b+1)}\right) \\ 0 \\ be^{bt}(t-t_0) \\ \beta\left(e^{tb} - e^{t+t_0(b-1)}\right) \end{bmatrix}
\end{aligned}
$$

where

$$\alpha = \frac{a}{b+1}, \beta = \frac{a^2b}{b-1}.$$

(c) (10 points) Compute $x(0)$, given that $x(2) = T\begin{bmatrix} a \\ 0 \\ 0 \\ -a^2b \end{bmatrix}$ and the control inputs are given in the previous question. You **do not have to simplify your answer. It should all be in terms of T, a, and b. It will look messy, and that's fine.**

Here, we have $t = 0$ and $t_0 = 2$, hence

$$
\begin{aligned}
x(t) &= T\begin{bmatrix} e^{t_0-t} & 0 & 0 & 0 \\ 0 & e^{a(t-t_0)} & 0 & 0 \\ 0 & 0 & e^{b(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{t-t_0} \end{bmatrix} T^{-1}x(t_0) + T\begin{bmatrix} \alpha\left(e^{bt} - e^{-t+t_0(b+1)}\right) \\ 0 \\ be^{bt}(t-t_0) \\ \beta\left(e^{tb} - e^{t+t_0(b-1)}\right) \end{bmatrix} \\
&= T\begin{bmatrix} e^{2} & 0 & 0 & 0 \\ 0 & e^{a(-2)} & 0 & 0 \\ 0 & 0 & e^{b(-2)} & 0 \\ 0 & 0 & 0 & e^{-2} \end{bmatrix} T^{-1}T\begin{bmatrix} a \\ 0 \\ 0 \\ -a^2b \end{bmatrix} + T\begin{bmatrix} \alpha\left(1 - e^{2(b+1)}\right) \\ 0 \\ -2b \\ \beta\left(1 - e^{2(b-1)}\right) \end{bmatrix} \\
&= T\begin{bmatrix} e^{2} & 0 & 0 & 0 \\ 0 & e^{a(-2)} & 0 & 0 \\ 0 & 0 & e^{b(-2)} & 0 \\ 0 & 0 & 0 & e^{-2} \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 0 \\ -a^2b \end{bmatrix} + T\begin{bmatrix} \alpha\left(1 - e^{2(b+1)}\right) \\ 0 \\ -2b \\ \beta\left(1 - e^{2(b-1)}\right) \end{bmatrix} \\
&= T\left(\begin{bmatrix} ae^2 \\ 0 \\ 0 \\ -a^2be^{-2} \end{bmatrix} + \begin{bmatrix} \alpha\left(1 - e^{2(b+1)}\right) \\ 0 \\ -2b \\ \beta\left(1 - e^{2(b-1)}\right) \end{bmatrix}\right) \\
x(0) &= T\begin{bmatrix} ae^2 + \frac{a}{b+1}\left(1 - e^{2(b+1)}\right) \\ 0 \\ -2b \\ -a^2be^{-2} + \frac{a^2b}{b-1}\left(1 - e^{2(b-1)}\right) \end{bmatrix}
\end{aligned}
$$

3. (25 total points) In the class, we discussed the prey-predator dynamical model of the interaction of two species. This model can be altered as follows:

$$\begin{aligned}\dot{x}_1(t) &= ax_1(t) - c\frac{x_1(t)x_2(t)}{1+d\cdot x_1(t)}\\ \dot{x}_2(t) &= -bx_2(t) + c\frac{x_1(t)x_2(t)}{1+d\cdot x_1(t)}\end{aligned}$$

to account for the saturation in the population of the prey, where $x_1(t)$ and $x_2(t)$ denote the population of the two species (prey and predator), and a, b, c, d are all positive constants.

(a) (10 points) Obtain the **nonzero equilibrium** solution for the above dynamical model. Your answer should be simplified. The equilibrium quantities x_1^e, x_2^e should be given in terms of a, b, c, d.

$$x^e = \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} \dfrac{b}{c-bd} \\ \dfrac{a}{c-bd} \end{bmatrix}.$$

(b) (10 points) Obtain the linearized representation for the above nonlinear system around the equilibrium point.

The linearized, state-space matrices can be written as:

$$A = \begin{bmatrix} a - cx_{2e}\dfrac{1}{(1+dx_{1e})^2} & -\dfrac{cx_{1e}}{1+dx_{1e}} \\ \dfrac{cx_{2e}}{(1+dx_{1e})^2} & -b + \dfrac{cx_{2e}}{(1+dx_{1e})^2} \end{bmatrix} =$$

$$\begin{bmatrix} a - c\dfrac{a}{c-bd}\cdot\dfrac{1}{(1+d\frac{b}{c-bd})^2} & -\dfrac{\frac{cb}{c-bd}}{1+d\frac{b}{c-bd}} \\ \dfrac{\frac{ca}{c-bd}}{(1+d\frac{b}{c-bd})^2} & -b + \dfrac{\frac{ca}{c-bd}}{(1+d\frac{b}{c-bd})^2} \end{bmatrix}.$$

(c) (5 points) Assume now that $a = 2$, $b = 4$, and $c = d = 2$. Is the linearized system obtained stable or unstable?

$$\begin{bmatrix} 2 - 2\frac{2}{2-8} \cdot \frac{1}{(1+2\frac{4}{2-8})^2} & -\frac{\frac{8}{2-8}}{1+2\frac{4}{2-8}} \\ \frac{\frac{4}{2-8}}{(1+2\frac{4}{2-8})^2} & -4 + \frac{\frac{4}{2-8}}{(1+2\frac{4}{2-8})^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -6 & -10 \end{bmatrix}.$$

The evalues of A are 9.2 and -11.2, which means A is has an evalue in the RHP, hence the linearized system around the given equilibrium point is **unstable.**

4. (20 total points) You are given the following CT LTI system

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Assume that the control input is constant between two sampling instances, i.e.,

$$u(t) = u(kT) =: u(k), \ \textbf{for} \ kT \leq t \leq (k+1)T, \ k = 0, 1, \ldots, k_f,$$

where T is the sampling time.

(a) (15 points) We wish to discretize the above continuous time system, and obtain:

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k).$$

Find the discretized state space matrices via the second discretization method we discussed in class—the more accurate one. You should derive this discretization method and explain all steps with thorough reasoning.

- Recall the solution to the state-equation:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Setting $t = KT$ in the previous equation, then we can write:

$$x(k) := x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

$$x(k+1) := x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

- Note that the above equation can be written as:

$$x(k+1) = e^{AT}\left(e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right)$$
$$+ \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau$$

- Recall that we're assuming that:

$$u(t) = u(kT) =: u(k) \qquad \text{for } kT \leq t \leq (k+1)T, \ k = 0, 1, \ldots, k_f$$

i.e., the input is constant between two sampling instances

- Look at $x(k)$ and let $\alpha = kT + T - \tau$, then:

$$x(k+1) = e^{AT}x(k) + \left(\int_0^T e^{A\alpha}d\alpha\right)Bu(k).$$

Hence,

$$\tilde{A} = e^{AT}, \tilde{B} = \left(\int_0^T e^{A\alpha}d\alpha\right)B.$$

(b) (5 points) Obtain $\tilde{A}, \tilde{B}$ given that

$$A = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T = 0.1.$$

$$\tilde{A} = e^{AT} = \begin{bmatrix} e^{0.1e} & 0 \\ 0 & 1 \end{bmatrix},$$

$$\tilde{B} = \left(\int_0^{0.1} e^{A\alpha} d\alpha \right) B = \int_0^{0.1} \begin{bmatrix} e^{e\alpha} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\alpha = \begin{bmatrix} e^{-1} \left(e^{0.1e} - 1 \right) \\ 0.1 \end{bmatrix}$$