Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 5 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	20	
2	20	
3	15	
4	20	
5	25	
Total	100	

1. (20 total points) Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

(a) (5 points) Given that $x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $u(k) = 0$ for all k, determine $x(15)$.

(b) (10 points) Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}$ and $x(0) = 0$.

(c) (5 points) Determine the stability of the system if $a = 0.8$ (asymptotically stable, marginally stable, or unstable?). What happens to $x(n)$ as $n \to \infty$?

2. (20 total points)

   (a) (15 points) a

   - a

   (b) (5 points) a

   s

3. (15 total points) Determine the stability of these systems (marginal, asymptotic, unstable). You have to clearly justify your answer.

   (a) (5 points)

$$x(k+1) = \begin{bmatrix} 0.995 & 0 \\ 0 & 2 \end{bmatrix} x(k)$$

> Unstable, since $eig(A) = \{0.4, 2\}$ and $\lambda_2 = 2 > 1$ is outside the unit disk.

   (b) (5 points)

$$\dot{x}(t) = T \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T^{-1} x(t) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} u(t)$$

> Marginally stable, since $eig(A) = \{-0.4, -0.4, -0.4, 0\}$. Hence, all eigenvalues of A are in the closed LHP, and the evalue on the $j\omega$ axis has Jordan block of size 1. Hence, the system is marginally stable.

   (c) (5 points)

$$\dot{x}(t) = -0.0000001 x(t) + 100000 u(t)$$

> System is asymptotically stable, as the only evalue for this system is $\lambda = -0.0000001$ which is still in the open LHP.

4. (20 total points) In Exam I, we found the equilibrium solution to prey-predator dynamical model of the interaction of two species

$$\dot{x}(t) = f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix}$$

where

$$\dot{x}_1(t) = f_1(x,u) = ax_1(t) - c\frac{x_1(t)x_2(t)}{1 + d \cdot x_1(t)} + u(t)x_1(t) \quad (1)$$

$$\dot{x}_2(t) = f_2(x,u) = -bx_2(t) + c\frac{x_1(t)x_2(t)}{1 + d \cdot x_1(t)} + u(t)x_2(t), \quad (2)$$

where $x_1(t)$ and $x_2(t)$ denote the population of the two species (prey and predator), $u(t)$ is a single input, and a, b, c, d are all positive constants. The equilibrium solution is:

$$x_{1e} = \frac{b}{c - bd}, \quad x_{2e} = \frac{a}{c - bd}, \quad u_e = 0.$$

(a) (20 points) Obtain the linearized representation for the above nonlinear system around the equilibrium points (x_e, u_e). Essentially, you have to obtain the linearized state space matrices:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$$

where $\tilde{x}(t) = x(t) - x_e$ and $\tilde{u}(t) = u(t) - u_e$. Notice that in this problem, **you only have one control input**. The state space matrices you obtain should be constants as a function of a, b, c, d.

The linearized, state-space matrices can be written as:

$$A = \begin{bmatrix} a + u_e - cx_{2e}\dfrac{1}{(1 + dx_{1e})^2} & -\dfrac{cx_{1e}}{1 + dx_{1e}} \\ \dfrac{cx_{2e}}{(1 + dx_{1e})^2} & -b + u_e + \dfrac{cx_{2e}}{(1 + dx_{1e})^2} \end{bmatrix} =$$

$$\begin{bmatrix} a - c\dfrac{a}{c - bd} \cdot \dfrac{1}{(1 + d\frac{b}{c - bd})^2} & -\dfrac{\frac{cb}{c - bd}}{1 + d\frac{b}{c - bd}} \\ \dfrac{\frac{ca}{c - bd}}{(1 + d\frac{b}{c - bd})^2} & -b + \dfrac{\frac{ca}{c - bd}}{(1 + d\frac{b}{c - bd})^2} \end{bmatrix}$$

$$B = \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} \dfrac{b}{c - bd} \\ \dfrac{a}{c - bd} \end{bmatrix}.$$

5. (25 total points) The following CT LTV system is given:

$$\dot{x}(t) = \begin{bmatrix} 3t^2 & \sin(t) & 0 \\ 0 & 3t^2 & 0 \\ 0 & 0 & 15\cos(0) \end{bmatrix} x(t).$$

(a) (20 points) Obtain the state transition matrix $\phi(t, t_0)$ for the above system. To receive full credit, you have to clearly show your steps. You do not have to multiply the individual matrices at the end of the STM computations.

- Notice that

$$A(t) = \begin{bmatrix} A_{11}(t) & \\ & A_{22}(t) \end{bmatrix}, A_{11}(t) = \begin{bmatrix} 3t^2 & \sin(t) \\ 0 & 3t^2 \end{bmatrix}, A_{22}(t) = \begin{bmatrix} 15\cos(0) = 15 \end{bmatrix}.$$

- Hence, the problem simplifies to finding two STMs for $A_{11}(t)$ and $A_{22}(t)$ since $A(t)$ is block diagonal.
- STM of $A_{22}(t)$:

$$\phi_{22}(t, t_0) = e^{15(t-t_0)}.$$

- STM of $A_{11}(t)$:
  Notice that

$$A_{11}(t) = 3t^2 I_2 + \sin(t) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent of order 2. Hence, we can write:

$$\phi_{22}(t, t_0) = e^{\int_{t_0}^{t} 3\tau^2 d\tau} \left(I_2 + (\cos(t_0) - \cos(t)) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = e^{t^3 - t_0^3} \begin{bmatrix} 1 & \cos(t_0) - \cos(t) \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \phi_A(t, t_0) = \begin{bmatrix} e^{t^3 - t_0^3} \begin{bmatrix} 1 & \cos(t_0) - \cos(t) \\ 0 & 1 \end{bmatrix} & \\ & e^{15(t-t_0)} \end{bmatrix}.$$

(b) (5 points) Find $x(t)$ if $x(t_0 = 2) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$$x(t) = \phi_A(t, t_0) x(t_0) = \begin{bmatrix} e^{t^3 - t_0^3} \begin{bmatrix} 1 & \cos(t_0) - \cos(t) \\ 0 & 1 \end{bmatrix} & \\ & e^{15(t-t_0)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} e^{t^3 - t_0^3} (1 + \cos(t) - \cos(t_0)) \\ -e^{t^3 - t_0^3} \\ 0 \end{bmatrix}$$